



Journal of Operations Intelligence

Journal homepage: www.jopi-journal.org
ISSN: 3009-4267

SCIENTIFIC OASIS



A Trapezoidal-Valued Fermatean Fuzzy Decision-Making Framework Using Aczel-Alsina Operators for Disaster Management

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ARTICLE INFO

Article history:

Received 31 July 2026

Received in revised form 2 September 2026

Accepted 28 September 2026

Available online 8 October 2026

Keywords:

Trapezoidal-valued Fermatean fuzzy sets; Aczel-Alsina operators; TOPSIS method; Disaster relief allocation.

ABSTRACT

This paper introduces trapezoidal-valued Fermatean fuzzy (TpVFF) numbers based on the Aczel-Alsina ($\mathcal{A}\mathcal{A}$) t-norm and t-conorm operations. New operational laws, including addition, multiplication, and scalar multiplication, are defined. Based on these operations, several geometric aggregation operators, including weighted, ordered weighted, and hybrid operators, are developed, and their fundamental properties are established. A TOPSIS-based multi-criteria group decision-making method is proposed within the developed framework. An illustrative disaster relief allocation problem using the TpVFF-TOPSIS framework demonstrates the applicability of the proposed approach. Sensitivity and Monte Carlo analyses are conducted to examine variations in the decision results under changes in the $\mathcal{A}\mathcal{A}$ parameter and criterion weights. The results indicate that the ranking is sensitive to changes in the $\mathcal{A}\mathcal{A}$ parameter and criterion weights, although it remains unchanged in a substantial proportion of the tested scenarios.

1. Introduction

Zadeh [1] first proposed the idea of fuzzy sets ($\mathcal{F}Ss$), thereby establishing a mathematical framework for representing uncertainty through partial membership. Atanassov [2] extended fuzzy set theory by introducing intuitionistic fuzzy sets ($\mathcal{IFS}s$), which characterize each element using membership and non-membership degrees. Later, Atanassov [3] provided additional operations to improve the mathematical structure of these sets. Yager [4] introduced Pythagorean fuzzy sets ($\mathcal{PFS}s$), which provide greater flexibility in representing uncertainty by relaxing the constraint on the member-

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<https://doi.org/10.31181/jopi41202685>

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ship and non-membership degrees. Fermatean fuzzy sets ($\mathcal{FFS}$ s) are an advanced generalization of $\mathcal{IFS}$ s and $\mathcal{PFS}$ s, where the sum of the cubes of the $\mathcal{M}$ and $\mathcal{NM}$ degrees is constrained, providing more flexibility in representing uncertainty, according to Senapati and Yager [5]. This framework provides greater flexibility for representing ambiguous and complex information in real-world decision-making problems. Trapezoidal-valued intuitionistic fuzzy numbers ($TpVIFNs$), a generalized form of real-valued and triangular intuitionistic fuzzy sets, were first described by Jeevaraj *et al.* [6].

1.1 Literature Review

Aggregation operators ($\mathcal{AO}$ s) have been widely developed to support decision-making ($\mathcal{DM}$) models in fuzzy environments. Xu [7] first presented geometric $\mathcal{AO}$ s for $\mathcal{IFS}$ s, which allow several criteria to be combined while preserving uncertainty information. Xu [8] developed multi-person models for multi-attribute decision-making that incorporate the perspectives of several decision makers. Zhao *et al.* [9] developed generalized $\mathcal{AO}$ s based on these foundations, thereby providing greater flexibility in managing various $\mathcal{IFS}$ s. Klement *et al.* [10] made a theoretical contribution by formalizing continuous triangular norms, which provide a mathematical foundation for the creation of consistent $\mathcal{AO}$ s. Biderci and Guneri [11] proposed a new risk assessment methodology utilizing $\mathcal{FFS}$ s. Their method provided an alternative way to represent uncertainty in risk assessment compared with $\mathcal{IFS}$ s and $\mathcal{PFS}$ s. For sustainable transportation, Singh *et al.* [12] developed a Fermatean fuzzy multi-objective indefinite quadratic transportation problem. $\mathcal{FFS}$ theory was used to represent competing objectives and ambiguous transportation factors. The method was illustrated through numerical examples involving sustainable transportation.

The extension of these operators to advanced fuzzy frameworks has been the focus of more recent efforts. To better capture interactions among criteria in decision-making, Senapati *et al.* [13, 14] introduced a series of Aczel–Alsina-based $\mathcal{AO}$ s using flexible t-norms and t-conorms. These concepts were expanded to bipolar complex fuzzy settings by Mahmood *et al.* [15], thereby enabling the simultaneous representation of positive and negative information in decision analysis. By permitting a wider range of $\mathcal{M}$ and $\mathcal{NM}$ values, the interval-valued Fermatean fuzzy $\mathcal{AA}$ geometric operators proposed by Meher and Jeevaraj [16] provide a more flexible representation of uncertainty. To help decision-makers represent hesitation, Naeem and Ali [17] proposed spherical fuzzy aggregation operators. Later, Ali [18] extended this line of research by applying T-spherical fuzzy sets to represent uncertainty more flexibly. Farid and Riaz [19] developed Aczel–Alsina-based $\mathcal{AO}$ s for q-rung orthopair fuzzy sets and established the mathematical properties of the geometric and weighted averaging operators. The approach was illustrated through an engineering decision-making problem.

Several studies have also demonstrated the practical applicability of these advanced models. In order to improve decision robustness in logistics service provider selection, Wang *et al.* [20] presented power $\mathcal{AO}$ s based on $\mathcal{AA}$ norms.

Additionally, Selvaraj *et al.* [21, 22] developed comparison and ranking procedures in decision analysis by creating new $\mathcal{AO}$ s and similarity metrics for trapezoidal-valued $\mathcal{IF}$ numbers. For generalized trapezoidal fuzzy numbers, Bihari *et al.* [23] introduced geometric and full ranking techniques, providing tools for applications such as supplier evaluation and security service selection. These studies demonstrate a progression from basic aggregation operators to more advanced $\mathcal{AA}$ -based models and their application to complex decision-making problems. Kaviyarasu *et al.* [24] introduced Dombi weighted geometric accumulation operators for $TpVFF$ numbers in $\mathcal{MAGDM}$ problems. These operators use Dombi-based operational laws for aggregation under uncertainty.

To represent complex and uncertain information in decision-making, this study extends previous research on $\mathcal{FFS}$ s by introducing $TpVFF$ numbers based on Aczel–Alsina t-norm and t-conorm operations. Based on these operational laws, including addition, multiplication, scalar multiplication, and power operations, we develop weighted, ordered weighted, and hybrid geometric aggregation operators and establish their idempotency, monotonicity, and boundedness properties. Additionally, we develop a TOPSIS-based $\mathcal{MCGDM}$ method within the $TpVFF$ - $\mathcal{AA}$ framework. Finally, an illustrative example of disaster relief allocation is presented to demonstrate the applicability of the proposed

method, along with a comparative analysis examining its consistency with an existing Dombi-based TpVFF approach.

1.2 Research gap and Motivation

The literature review indicates that $\mathcal{AO}s$ using $\mathcal{AA}$ t-norms and t-conorms have been proposed for many fuzzy sets such as $\mathcal{IFS}s$, interval-valued $\mathcal{FFS}s$, spherical $\mathcal{FS}s$, T-spherical $\mathcal{FS}s$, and q-rung orthopair $\mathcal{FS}s$. However, to the best of our knowledge, $\mathcal{AA}$ -based aggregation operators have not yet been developed for the TpVFF environment, which represents uncertain information through trapezoidal-valued membership and non-membership degrees. This motivates the development of $\mathcal{AA}$ operational laws for TpVFF numbers. Secondly, in our previous study, Kaviyarasu *et al.* [24] proposed Dombi weighted geometric aggregation operators for TpVFF numbers. However, $\mathcal{AA}$ aggregation operators employ parameterized aggregation mechanisms that have not yet been defined for this environment. Therefore, we propose $TpVFFAAWG$, $TpVFFAAOWG$, and $TpVFFAAHG$ operators along with their corresponding laws and properties. Finally, we develop a TpVFF-TOPSIS framework that incorporates the proposed $\mathcal{AA}$ aggregation operators to address $\mathcal{MCGDM}$ problems involving TpVFF information. The proposed framework is demonstrated through a hypothetical disaster relief allocation problem.

1.3 Contribution of the Study

The primary goal of this study is to propose $\mathcal{AA}$ -based geometric operators for TpVFF numbers and investigate their application in $\mathcal{MCGDM}$. The proposed framework provides a way to represent uncertainty within the TpVFF environment. The primary contributions of this study are as follows:

- We introduce $\mathcal{AA}$ -based operational laws for TpVFF numbers, including addition, multiplication, and scalar multiplication.
- We develop several TpVFF geometric $\mathcal{AO}s$, including $TpVFFAAWG$, $TpVFFAAOWG$ and $TpVFFAAHG$.
- Through suitable theorems, we establish important mathematical properties of the proposed $\mathcal{AO}s$, including idempotency, monotonicity, and boundedness.
- We develop a TpVFF-TOPSIS method based on the proposed aggregation operators for $\mathcal{MCGDM}$.
- We use a hypothetical disaster relief allocation problem to illustrate the applicability of the suggested approach.
- Furthermore, sensitivity and Monte Carlo analyses are conducted to examine variations in the decision results with respect to the $\mathcal{AA}$ parameter and criterion weights.

1.4 Organisation of the Paper

Section 2 provides the essential definitions, key ideas, and scoring functions for TpVFF numbers. Section 3 presents the $\mathcal{AA}$ -based operational laws for TpVFF numbers and their characteristics. Section 4 introduces the new $\mathcal{AO}s$, namely, $TpVFFAAWG$, $TpVFFAAOWG$, and $TpVFFAAHG$, and presents theorems to investigate their characteristics, including idempotency, boundedness, and monotonicity. Section 5 presents a modified TpVFF-TOPSIS approach for $\mathcal{MCGDM}$ problems using the proposed aggregation operators. An illustrative disaster relief allocation problem is then provided to demonstrate its applicability. Section 6 presents the sensitivity analysis for different values of the $\mathcal{AA}$ parameter. Section 7 presents a comparative analysis with an existing Dombi-based approach and discusses the reproducibility of the calculations and the sensitivity of the results. Finally, the conclusions and future research directions are presented at the end of the paper.

2. Preliminaries

Definition 1. [2] Let $\mathcal{V}_{t_p}$ be a non-empty set. An $\mathcal{IFS}$ $\mathcal{I}_{t_p}$ is characterized by

$$\mathcal{I}_{t_p} = \{\langle k_{t_p}, a_{\mathcal{I}_{t_p}}(k_{t_p}), b_{\mathcal{I}_{t_p}}(k_{t_p}) \rangle | k_{t_p} \in \mathcal{V}_{t_p}\},$$

where $a_{\mathcal{I}_{t_p}} : \mathcal{V}_{t_p} \rightarrow [0, 1]$ and $b_{\mathcal{I}_{t_p}} : \mathcal{V}_{t_p} \rightarrow [0, 1]$, $k_{t_p} \in \mathcal{V}_{t_p}$ with the conditions $0 \leq a_{\mathcal{I}_{t_p}}(k_{t_p}) + b_{\mathcal{I}_{t_p}}(k_{t_p}) \leq 1$. The values $a_{\mathcal{I}_{t_p}}(k_{t_p})$ and $b_{\mathcal{I}_{t_p}}(k_{t_p})$ represent the membership and non-membership degrees, respectively.

Definition 2. [5] Let $\mathcal{V}_{t_p}$ be a non-empty set. An $\mathcal{FFS}$ $\mathcal{F}_{t_p}$ is characterized by

$$\mathcal{F}_{t_p} = \{ \langle k_{t_p}, a_{\mathcal{F}_{t_p}}(k_{t_p}), b_{\mathcal{F}_{t_p}}(k_{t_p}) \rangle | k_{t_p} \in \mathcal{V}_{t_p} \},$$

where $a_{\mathcal{F}_{t_p}} : \mathcal{V}_{t_p} \rightarrow [0, 1]$ and $b_{\mathcal{F}_{t_p}} : \mathcal{V}_{t_p} \rightarrow [0, 1]$, $k_{t_p} \in \mathcal{V}_{t_p}$ with the conditions $0 \leq a_{\mathcal{F}_{t_p}}(k_{t_p})^3 + b_{\mathcal{F}_{t_p}}(k_{t_p})^3 \leq 1$. The values $a_{\mathcal{F}_{t_p}}(k_{t_p})$ and $b_{\mathcal{F}_{t_p}}(k_{t_p})$ represent the membership and non-membership degrees, respectively.

Based on the concepts of Fermatean fuzzy sets and trapezoidal fuzzy numbers [5, 6], we define a trapezoidal-valued Fermatean fuzzy number as follows.

Definition 3. Let $\hat{T}_p = \langle (\hat{a}_p, \hat{b}_p, \hat{c}_p, \hat{d}_p), (\hat{w}_p, \hat{x}_p, \hat{y}_p, \hat{z}_p) \rangle$ be a TpVFF number, where the $\mathcal{M}$ and $\mathcal{NM}$ components satisfy $0 \leq \hat{a}_p \leq \hat{b}_p \leq \hat{c}_p \leq \hat{d}_p \leq 1$, and $0 \leq \hat{w}_p \leq \hat{x}_p \leq \hat{y}_p \leq \hat{z}_p \leq 1$. Furthermore, the Fermatean condition is satisfied componentwise, namely

$$\begin{aligned} \hat{a}_p^3 + \hat{w}_p^3 &\leq 1, & \hat{b}_p^3 + \hat{x}_p^3 &\leq 1, \\ \hat{c}_p^3 + \hat{y}_p^3 &\leq 1, & \hat{d}_p^3 + \hat{z}_p^3 &\leq 1. \end{aligned}$$

Here, $(\hat{a}^p, \hat{b}^p, \hat{c}^p, \hat{d}^p)$ and $(\hat{w}^p, \hat{x}^p, \hat{y}^p, \hat{z}^p)$ denote the trapezoidal membership and non-membership parameters, respectively. For convenience, a TpVFF number is denoted by

$$\hat{T}_p = \langle (\hat{a}_p, \hat{b}_p, \hat{c}_p, \hat{d}_p), (\hat{w}_p, \hat{x}_p, \hat{y}_p, \hat{z}_p) \rangle.$$

The trapezoidal membership parameters are consistently represented in ascending order, i.e., $\hat{a}_p \leq \hat{b}_p \leq \hat{c}_p \leq \hat{d}_p$, and $\hat{w}_p \leq \hat{x}_p \leq \hat{y}_p \leq \hat{z}_p$.

Definition 4. [10] A t-norm ($\bar{T}$) is a function from $[0, 1] \times [0, 1]$ to $[0, 1]$ that satisfies the following conditions for all $g_1, g_2, g_3 \in [0, 1]$

1. Symmetric: $\bar{T}(g_1, g_2) = \bar{T}(g_2, g_1)$
2. Monotonic: $\bar{T}(g_1, g_2) \leq \bar{T}(g_1, g_3)$ if $g_2 \leq g_3$
3. Associative: $\bar{T}(g_1, \bar{T}(g_2, g_3)) = \bar{T}(\bar{T}(g_1, g_2), g_3)$
4. Identity: $\bar{T}(g_1, 1) = g_1$

Definition 5. [10] A t-conorm ($\bar{S}$) is a function from $[0, 1] \times [0, 1]$ to $[0, 1]$ that satisfies the following conditions for all $g_1, g_2, g_3 \in [0, 1]$

1. Symmetric: $\bar{S}(g_1, g_2) = \bar{S}(g_2, g_1)$
2. Monotonic: $\bar{S}(g_1, g_2) \leq \bar{S}(g_1, g_3)$ if $g_2 \leq g_3$
3. Associative: $\bar{S}(g_1, \bar{S}(g_2, g_3)) = \bar{S}(\bar{S}(g_1, g_2), g_3)$
4. Identity: $\bar{S}(g_1, 0) = g_1$

Definition 6. [25] and [26] The Aczel-Alsina t-norm ($\bar{T}_{\mathbb{Z}}^A$) $_{\mathbb{Z} \in (0, \infty]}$ is defined as follows:

$$\bar{T}_{\mathbb{Z}}^A(g_1, g_2) = \begin{cases} \min(g_1, g_2), & \text{if } \mathbb{Z} = \infty \\ e^{-((- \ln g_1)^{\mathbb{Z}} + (- \ln g_2)^{\mathbb{Z}})^{\frac{1}{\mathbb{Z}}}}, & \text{if } 0 < \mathbb{Z} < \infty \end{cases}$$

The Aczel-Alsina t-conorm ($\bar{S}_{\mathbb{Z}}^A$) $_{\mathbb{Z} \in (0, \infty]}$ is defined as follows:

$$\bar{S}_{\mathbb{Z}}^A(g_1, g_2) = \begin{cases} \max(g_1, g_2), & \text{if } \mathbb{Z} = \infty \\ 1 - e^{-((- \ln(1-g_1))^{\mathbb{Z}} + (- \ln(1-g_2))^{\mathbb{Z}})^{\frac{1}{\mathbb{Z}}}}, & \text{if } 0 < \mathbb{Z} < \infty \end{cases}$$

For $0 < \mathbb{Z} < \infty$, the explicit $\mathcal{AA}$ expressions involving $1/\mathbb{Z}$ are used. For $\mathbb{Z} = \infty$, the t-norm and t-conorm reduce to the minimum and maximum operators, respectively. The value $\mathbb{Z} = 0$ is excluded from the admissible parameter domain.

Definition 7. [27] Let $T_p = \langle (a_{\hat{a}}, b_{\hat{a}}, c_{\hat{a}}, d_{\hat{a}}), (w_{\hat{b}}, x_{\hat{b}}, y_{\hat{b}}, z_{\hat{b}}) \rangle$ be a TpVFF number. Then, the score function S is defined as follows:

$$S = \frac{\{[a_{\hat{a}}^3 + b_{\hat{a}}^3 + c_{\hat{a}}^3 + d_{\hat{a}}^3]\} - [w_{\hat{b}}^3 + x_{\hat{b}}^3 + y_{\hat{b}}^3 + z_{\hat{b}}^3]}{8}$$

3. Aczel-Alsina t-norm and t-conorm

This section introduces the fundamental arithmetic operations for TpVFF numbers, including the TpVFF sum, product, scalar multiplication, and power operations.

Definition 8. Let T_{p1} and T_{p2} be two TpVFF numbers, and let $Z \in (0, \infty]$ be the Aczel–Alsina parameter. The TpVFF sum and product operations are defined as follows:

(i) $T_{p1} \oplus T_{p2}$

$$= \left\langle \left(\sqrt[3]{1 - e^{-\left((-\ln(1-a_{1a}^3))^Z + (-\ln(1-a_{2a}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((-\ln(1-b_{1a}^3))^Z + (-\ln(1-b_{2a}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \right. \\ \left. \sqrt[3]{1 - e^{-\left((-\ln(1-c_{1a}^3))^Z + (-\ln(1-c_{2a}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((-\ln(1-d_{1a}^3))^Z + (-\ln(1-d_{2a}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\ \left. \left(e^{-\left((-\ln(w_{1b}))^Z + (-\ln(w_{2b}))^Z \right)^{\frac{1}{Z}}}, e^{-\left((-\ln(x_{1b}))^Z + (-\ln(x_{2b}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\ \left. \left. e^{-\left((-\ln(y_{1b}))^Z + (-\ln(y_{2b}))^Z \right)^{\frac{1}{Z}}}, e^{-\left((-\ln(z_{1b}))^Z + (-\ln(z_{2b}))^Z \right)^{\frac{1}{Z}}} \right) \right\rangle$$

(ii) $T_{p1} \otimes T_{p2} =$

$$= \left\langle \left(e^{-\left((-\ln(a_{1a}))^Z + (-\ln(a_{2a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left((-\ln(b_{1a}))^Z + (-\ln(b_{2a}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\ \left. \left. e^{-\left((-\ln(c_{1a}))^Z + (-\ln(c_{2a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left((-\ln(d_{1a}))^Z + (-\ln(d_{2a}))^Z \right)^{\frac{1}{Z}}} \right), \right. \\ \left(\sqrt[3]{1 - e^{-\left((-\ln(1-w_{1b}^3))^Z + (-\ln(1-w_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((-\ln(1-x_{1b}^3))^Z + (-\ln(1-x_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\ \left. \sqrt[3]{1 - e^{-\left((-\ln(1-y_{1b}^3))^Z + (-\ln(1-y_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((-\ln(1-z_{1b}^3))^Z + (-\ln(1-z_{2b}^3))^Z \right)^{\frac{1}{Z}}}} \right) \right\rangle$$

(iii) For $\zeta > 0$, scalar multiplication of a TpVFF number is defined as follows: ζT_p

$$= \left\langle \left(\sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-a_{\hat{a}}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-b_{\hat{a}}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \right. \\ \left. \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-c_{\hat{a}}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-d_{\hat{a}}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\ \left. \left(e^{-\left(\zeta(-\ln(w_{\hat{b}}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(x_{\hat{b}}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(y_{\hat{b}}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(z_{\hat{b}}))^Z \right)^{\frac{1}{Z}}} \right) \right\rangle$$

(iv) For $\zeta > 0$,

$$\begin{aligned} T_p^\zeta = & \left\langle \left(e^{-\left(\zeta(-\ln(a_{\hat{a}}))^{\mathbb{Z}}\right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\zeta(-\ln(b_{\hat{a}}))^{\mathbb{Z}}\right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\zeta(-\ln(c_{\hat{a}}))^{\mathbb{Z}}\right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\zeta(-\ln(d_{\hat{a}}))^{\mathbb{Z}}\right)^{\frac{1}{\mathbb{Z}}}} \right), \right. \\ & \left(\sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-w_b^3))^{\mathbb{Z}}\right)^{\frac{1}{\mathbb{Z}}}}}, \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-x_b^3))^{\mathbb{Z}}\right)^{\frac{1}{\mathbb{Z}}}}}, \right. \\ & \left. \left. \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-y_b^3))^{\mathbb{Z}}\right)^{\frac{1}{\mathbb{Z}}}}}, \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-z_b^3))^{\mathbb{Z}}\right)^{\frac{1}{\mathbb{Z}}}}} \right) \right\rangle \end{aligned}$$

(v) For $\zeta = 0$, scalar multiplication is defined separately by

$$0T_p = \langle (0, 0, 0, 0), (1, 1, 1, 1) \rangle.$$

This convention avoids direct evaluation of the logarithmic expressions in Definition 8(iii) at $\zeta = 0$, including cases in which a component equals 0 or 1.

Lemma 1. Let $T_p = \langle (a_{\hat{a}}, b_{\hat{a}}, c_{\hat{a}}, d_{\hat{a}}), (w_{\hat{b}}, x_{\hat{b}}, y_{\hat{b}}, z_{\hat{b}}) \rangle$ be a TpVFF number satisfying $0 \leq a_{\hat{a}} \leq b_{\hat{a}} \leq c_{\hat{a}} \leq d_{\hat{a}} \leq 1$, $0 \leq w_{\hat{b}} \leq x_{\hat{b}} \leq y_{\hat{b}} \leq z_{\hat{b}} \leq 1$, and $a_{\hat{a}}^3 + w_{\hat{b}}^3 \leq 1$, $b_{\hat{a}}^3 + x_{\hat{b}}^3 \leq 1$, $c_{\hat{a}}^3 + y_{\hat{b}}^3 \leq 1$, $d_{\hat{a}}^3 + z_{\hat{b}}^3 \leq 1$. Then, for $\zeta > 0$, the scalar multiplication ζT_p defined in Definition 8(iii) is also a TpVFF number.

Proof. First, consider $0 < \mathbb{Z} < \infty$ and $\zeta > 0$. From Definition 8 (iii), the scalar multiplication can be rewritten as $a'_a = \left[1 - (1 - a_{\hat{a}}^3)^{\zeta^{1/\mathbb{Z}}}\right]^{1/3}$, $w'_b = w_{\hat{b}}^{\zeta^{1/\mathbb{Z}}}$, and similarly, $b'_a = \left[1 - (1 - b_{\hat{a}}^3)^{\zeta^{1/\mathbb{Z}}}\right]^{1/3}$, $x'_b = x_{\hat{b}}^{\zeta^{1/\mathbb{Z}}}$, $c'_a = \left[1 - (1 - c_{\hat{a}}^3)^{\zeta^{1/\mathbb{Z}}}\right]^{1/3}$, $y'_b = y_{\hat{b}}^{\zeta^{1/\mathbb{Z}}}$, and $d'_a = \left[1 - (1 - d_{\hat{a}}^3)^{\zeta^{1/\mathbb{Z}}}\right]^{1/3}$, $z'_b = z_{\hat{b}}^{\zeta^{1/\mathbb{Z}}}$. Since $\zeta > 0$ and $\mathbb{Z} > 0$, we have $\zeta^{1/\mathbb{Z}} > 0$. Hence, the functions $f(t) = \left[1 - (1 - t^3)^{\zeta^{1/\mathbb{Z}}}\right]^{1/3}$ and $g(t) = t^{\zeta^{1/\mathbb{Z}}}$ are nondecreasing on $[0, 1]$. Therefore, $a'_a \leq b'_a \leq c'_a \leq d'_a$, and $w'_b \leq x'_b \leq y'_b \leq z'_b$. Moreover, all transformed components belong to $[0, 1]$. It remains to verify that the Fermatean condition is satisfied. Since $a_{\hat{a}}^3 + w_{\hat{b}}^3 \leq 1$, we have $w_{\hat{b}}^3 \leq 1 - a_{\hat{a}}^3$. Because $\zeta^{1/\mathbb{Z}} > 0$, it follows that $(w'_b)^3 = w_{\hat{b}}^{3\zeta^{1/\mathbb{Z}}} \leq (1 - a_{\hat{a}}^3)^{\zeta^{1/\mathbb{Z}}}$. Hence, $(a'_a)^3 + (w'_b)^3 = 1 - (1 - a_{\hat{a}}^3)^{\zeta^{1/\mathbb{Z}}} + w_{\hat{b}}^{3\zeta^{1/\mathbb{Z}}} \leq 1$. Similarly, $(b'_a)^3 + (x'_b)^3 \leq 1$, $(c'_a)^3 + (y'_b)^3 \leq 1$, and $(d'_a)^3 + (z'_b)^3 \leq 1$. Thus, the Fermatean condition is preserved for $0 < \mathbb{Z} < \infty$. For $\mathbb{Z} = \infty$, the $\mathcal{AA}$ t-conorm and t-norm reduce to the maximum and minimum operators, respectively. Therefore, from Definition 8 (iii), $a'_a = [1 - \max\{1 - a_{\hat{a}}^3, 0\}]^{1/3} = a_{\hat{a}}$, $w'_b = \min\{w_{\hat{b}}, 1\} = w_{\hat{b}}$, and similarly, $b'_a = b_{\hat{a}}$, $c'_a = c_{\hat{a}}$, $d'_a = d_{\hat{a}}$, and $x'_b = x_{\hat{b}}$, $y'_b = y_{\hat{b}}$, $z'_b = z_{\hat{b}}$. Thus, the scalar multiplication preserves the trapezoidal ordering and the Fermatean condition for $\mathbb{Z} = \infty$ as well. Therefore, for all $\mathbb{Z} \in (0, \infty]$, including the limiting case $\mathbb{Z} = \infty$, ζT_p is a TpVFF number. $\square$

Lemma 2. Let $T_{p_1} = \langle (a_{\hat{1}a}, b_{\hat{1}a}, c_{\hat{1}a}, d_{\hat{1}a}), (w_{\hat{1}b}, x_{\hat{1}b}, y_{\hat{1}b}, z_{\hat{1}b}) \rangle$ and $T_{p_2} = \langle (a_{\hat{2}a}, b_{\hat{2}a}, c_{\hat{2}a}, d_{\hat{2}a}), (w_{\hat{2}b}, x_{\hat{2}b}, y_{\hat{2}b}, z_{\hat{2}b}) \rangle$ be two TpVFF numbers belonging to the set Ω of all admissible TpVFF numbers. Then, for $\mathbb{Z} > 0$, the TpVFF-sum defined in Definition 8(i) satisfies $T_{p_1} \oplus T_{p_2} \in \Omega$.

Proof. For $0 < \mathbb{Z} < \infty$, denote the resulting $\mathcal{M}$ components of $T_{p_1} \oplus T_{p_2}$ by a'_a, b'_a, c'_a, d'_a and the corresponding $\mathcal{NM}$ components by w'_b, x'_b, y'_b, z'_b . Because the $\mathcal{AA}$ t-conorm and t-norm map $[0, 1]^2$ into $[0, 1]$, all resulting components lie in $[0, 1]$. Moreover, both the $\mathcal{AA}$ t-conorm and t-norm are nondecreasing in each argument. Hence, from $a_{\hat{1}a} \leq b_{\hat{1}a} \leq c_{\hat{1}a} \leq d_{\hat{1}a}$, $a_{\hat{2}a} \leq b_{\hat{2}a} \leq c_{\hat{2}a} \leq d_{\hat{2}a}$, the $\mathcal{M}$ components of the resulting number preserve the trapezoidal ordering $a'_a \leq b'_a \leq c'_a \leq d'_a$. Similarly, $w'_b \leq x'_b \leq y'_b \leq z'_b$.

It remains to verify the Fermatean condition. For the first pair, let $A_1 = a_{\hat{1}a}^3$, $A_2 = a_{\hat{2}a}^3$, $W_1 = w_{\hat{1}b}^3$, $W_2 = w_{\hat{2}b}^3$. Since T_{p_1} and T_{p_2} are admissible, $W_1 \leq 1 - A_1$, $W_2 \leq 1 - A_2$. By the monotonicity of the Aczel-Alsina t-norm, $T_{\mathcal{AA}}(W_1, W_2) \leq T_{\mathcal{AA}}(1 - A_1, 1 - A_2)$. By the duality between the Aczel-Alsina

t -norm and t -conorm, $S_{\mathcal{AA}}(A_1, A_2) + T_{\mathcal{AA}}(1 - A_1, 1 - A_2) = 1$. Therefore, $(a'_a)^3 + (w'_b)^3 \leq 1$. Applying the same argument to the remaining three corresponding pairs gives $(b'_a)^3 + (x'_b)^3 \leq 1$, $(c'_a)^3 + (y'_b)^3 \leq 1$, and $(d'_a)^3 + (z'_b)^3 \leq 1$. For $\mathbb{Z} = \infty$, the Aczel-Alsina t -conorm and t -norm reduce to the maximum and minimum operators, respectively. Hence, $a'_a = \max\{a_{1a}, a_{2a}\}$, $w'_b = \min\{w_{1b}, w_{2b}\}$. Since $w_{1b}^3 \leq 1 - a_{1a}^3$, $w_{2b}^3 \leq 1 - a_{2a}^3$, we have $\min\{w_{1b}^3, w_{2b}^3\} \leq 1 - \max\{a_{1a}^3, a_{2a}^3\}$. Thus, $(a'_a)^3 + (w'_b)^3 \leq 1$. The same argument applies to the remaining three pairs. Therefore, the resulting components satisfy the boundedness condition, trapezoidal ordering, and all four componentwise Fermatean conditions. Hence, $T_{p_1} \oplus T_{p_2} \in \Omega$. Thus, the $TpVFF$ sum defined in Definition 8(i) is closed on Ω . $\square$

Lemma 3. Let $T_{p_1} = \langle (a_{1a}, b_{1a}, c_{1a}, d_{1a}), (w_{1b}, x_{1b}, y_{1b}, z_{1b}) \rangle$ and $T_{p_2} = \langle (a_{2a}, b_{2a}, c_{2a}, d_{2a}), (w_{2b}, x_{2b}, y_{2b}, z_{2b}) \rangle$ be two $TpVFF$ numbers belonging to the set Ω of all admissible $TpVFF$ numbers. Then, for $\mathbb{Z} > 0$, the $TpVFF$ -product defined in Definition 8(ii) satisfies $T_{p_1} \otimes T_{p_2} \in \Omega$.

Proof. For $0 < \mathbb{Z} < \infty$, denote the resulting $\mathcal{M}$ components of $T_{p_1} \otimes T_{p_2}$ by a'_a, b'_a, c'_a, d'_a and the corresponding $\mathcal{NM}$ components by w'_b, x'_b, y'_b, z'_b .

Since the $\mathcal{AA}$ t -norm and t -conorm map $[0, 1]^2$ into $[0, 1]$, all resulting components belong to $[0, 1]$. Because these operators are nondecreasing in each argument, they preserve the trapezoidal ordering of the membership and non-membership components. In particular, $a'_a \leq b'_a \leq c'_a \leq d'_a$, and $w'_b \leq x'_b \leq y'_b \leq z'_b$. It remains to verify the Fermatean condition. For the first pair, let $A_1 = a_{1a}^3, A_2 = a_{2a}^3, W_1 = w_{1b}^3, W_2 = w_{2b}^3$. The admissibility of T_{p_1} and T_{p_2} gives $W_1 \leq 1 - A_1, W_2 \leq 1 - A_2$. By the monotonicity of the $\mathcal{AA}$ t -conorm, $S_{\mathcal{AA}}(W_1, W_2) \leq S_{\mathcal{AA}}(1 - A_1, 1 - A_2)$. Using the duality relation, $S_{\mathcal{AA}}(1 - A_1, 1 - A_2) = 1 - T_{\mathcal{AA}}(A_1, A_2)$. Consequently, $(a'_a)^3 + (w'_b)^3 \leq 1$. Similarly, $(b'_a)^3 + (x'_b)^3 \leq 1$, $(c'_a)^3 + (y'_b)^3 \leq 1$, and $(d'_a)^3 + (z'_b)^3 \leq 1$. For $\mathbb{Z} = \infty$, the $\mathcal{AA}$ t -norm and t -conorm reduce to the minimum and maximum operators, respectively. Hence, $a'_a = \min\{a_{1a}, a_{2a}\}$, $w'_b = \max\{w_{1b}, w_{2b}\}$. Since $w_{1b}^3 \leq 1 - a_{1a}^3, w_{2b}^3 \leq 1 - a_{2a}^3$, we obtain $\max\{w_{1b}^3, w_{2b}^3\} \leq 1 - \min\{a_{1a}^3, a_{2a}^3\}$. Therefore, $(a'_a)^3 + (w'_b)^3 \leq 1$. The same argument applies to the remaining three pairs. Thus, the resulting $TpVFF$ satisfies the boundedness condition, trapezoidal ordering, and all four componentwise Fermatean conditions. Hence, $T_{p_1} \otimes T_{p_2} \in \Omega$. Therefore, the $TpVFF$ product defined in Definition 8(ii) is closed on Ω . $\square$

Theorem 1. For any $TpVFF$ numbers $T_p = \langle (a_a, b_a, c_a, d_a), (w_b, x_b, y_b, z_b) \rangle$, $T_{p_1} = \langle (a_{1a}, b_{1a}, c_{1a}, d_{1a}), (w_{1b}, x_{1b}, y_{1b}, z_{1b}) \rangle$, and $T_{p_2} = \langle (a_{2a}, b_{2a}, c_{2a}, d_{2a}), (w_{2b}, x_{2b}, y_{2b}, z_{2b}) \rangle$, where $\zeta, \zeta_1, \zeta_2 > 0$ are scalar parameters, the following properties hold:

- (i) $T_{p_1} \oplus T_{p_2} = T_{p_2} \oplus T_{p_1}$
- (ii) $T_{p_1} \otimes T_{p_2} = T_{p_2} \otimes T_{p_1}$
- (iii) $\zeta(T_{p_1} \oplus T_{p_2}) = \zeta T_{p_1} \oplus \zeta T_{p_2}$
- (iv) $(\zeta_1 + \zeta_2)T_p = \zeta_1 T_p \oplus \zeta_2 T_p$
- (v) $(T_{p_1} \otimes T_{p_2})^\zeta = (T_{p_1}^\zeta \otimes T_{p_2}^\zeta)$
- (vi) $T_p^{\zeta_1} \otimes T_p^{\zeta_2} = T_p^{\zeta_1 + \zeta_2}$

Proof. (i) $T_{p_1} \oplus T_{p_2}$

$$\begin{aligned}
&= \left\langle \left(\sqrt[3]{1 - \mathbf{e}^{-\left((- \ln(1 - \mathbf{a}_{1a}^3))^{\mathbb{Z}} + (- \ln(1 - \mathbf{a}_{2a}^3))^{\mathbb{Z}}\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left((- \ln(1 - \mathbf{b}_{1a}^3))^{\mathbb{Z}} + (- \ln(1 - \mathbf{b}_{2a}^3))^{\mathbb{Z}}\right)}}, \right. \\
&\quad \sqrt[3]{1 - \mathbf{e}^{-\left((- \ln(1 - \mathbf{c}_{1a}^3))^{\mathbb{Z}} + (- \ln(1 - \mathbf{c}_{2a}^3))^{\mathbb{Z}}\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left((- \ln(1 - \mathbf{d}_{1a}^3))^{\mathbb{Z}} + (- \ln(1 - \mathbf{d}_{2a}^3))^{\mathbb{Z}}\right)}}, \\
&\quad \left(\mathbf{e}^{-\left((- \ln(\mathbf{w}_{1b}))^{\mathbb{Z}} + (- \ln(\mathbf{w}_{2b}))^{\mathbb{Z}}\right)} \right)^{\frac{1}{\mathbb{Z}}}, \mathbf{e}^{-\left((- \ln(\mathbf{x}_{1b}))^{\mathbb{Z}} + (- \ln(\mathbf{x}_{2b}))^{\mathbb{Z}}\right)} \right)^{\frac{1}{\mathbb{Z}}}, \\
&\quad \left. \mathbf{e}^{-\left((- \ln(\mathbf{y}_{1b}))^{\mathbb{Z}} + (- \ln(\mathbf{y}_{2b}))^{\mathbb{Z}}\right)} \right)^{\frac{1}{\mathbb{Z}}}, \mathbf{e}^{-\left((- \ln(\mathbf{z}_{1b}))^{\mathbb{Z}} + (- \ln(\mathbf{z}_{2b}))^{\mathbb{Z}}\right)} \right)^{\frac{1}{\mathbb{Z}}} \rangle \\
&= \left\langle \left(\sqrt[3]{1 - \mathbf{e}^{-\left((- \ln(1 - \mathbf{a}_{2a}^3))^{\mathbb{Z}} + (- \ln(1 - \mathbf{a}_{1a}^3))^{\mathbb{Z}}\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left((- \ln(1 - \mathbf{b}_{2a}^3))^{\mathbb{Z}} + (- \ln(1 - \mathbf{b}_{1a}^3))^{\mathbb{Z}}\right)}}, \right. \\
&\quad \sqrt[3]{1 - \mathbf{e}^{-\left((- \ln(1 - \mathbf{c}_{2a}^3))^{\mathbb{Z}} + (- \ln(1 - \mathbf{c}_{1a}^3))^{\mathbb{Z}}\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left((- \ln(1 - \mathbf{d}_{2a}^3))^{\mathbb{Z}} + (- \ln(1 - \mathbf{d}_{1a}^3))^{\mathbb{Z}}\right)}}, \\
&\quad \left(\mathbf{e}^{-\left((- \ln(\mathbf{w}_{2b}))^{\mathbb{Z}} + (- \ln(\mathbf{w}_{1b}))^{\mathbb{Z}}\right)} \right)^{\frac{1}{\mathbb{Z}}}, \mathbf{e}^{-\left((- \ln(\mathbf{x}_{2b}))^{\mathbb{Z}} + (- \ln(\mathbf{x}_{1b}))^{\mathbb{Z}}\right)} \right)^{\frac{1}{\mathbb{Z}}}, \\
&\quad \left. \mathbf{e}^{-\left((- \ln(\mathbf{y}_{2b}))^{\mathbb{Z}} + (- \ln(\mathbf{y}_{1b}))^{\mathbb{Z}}\right)} \right)^{\frac{1}{\mathbb{Z}}}, \mathbf{e}^{-\left((- \ln(\mathbf{z}_{2b}))^{\mathbb{Z}} + (- \ln(\mathbf{z}_{1b}))^{\mathbb{Z}}\right)} \right)^{\frac{1}{\mathbb{Z}}} \rangle \\
&= T_{p_2} \oplus T_{p_1}
\end{aligned}$$

$$(ii) T_{p_1} \otimes T_{p_2}$$

$$\begin{aligned}
&= \left\langle \left(\mathbf{e}^{-\left((-\ln(a_{1a}))^Z + (-\ln(a_{2a}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left((-\ln(b_{1a}))^Z + (-\ln(b_{2a}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\
&\quad \left. \mathbf{e}^{-\left((-\ln(c_{1a}))^Z + (-\ln(c_{2a}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left((-\ln(d_{1a}))^Z + (-\ln(d_{2a}))^Z \right)^{\frac{1}{Z}}} \right), \\
&\quad \left(\sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-w_{1b}^3))^Z + (-\ln(1-w_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-x_{1b}^3))^Z + (-\ln(1-x_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\
&\quad \left. \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-y_{1b}^3))^Z + (-\ln(1-y_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-z_{1b}^3))^Z + (-\ln(1-z_{2b}^3))^Z \right)^{\frac{1}{Z}}}}} \right) \rangle \\
&= \left\langle \left(\mathbf{e}^{-\left((-\ln(a_{2a}))^Z + (-\ln(a_{1a}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left((-\ln(b_{2a}))^Z + (-\ln(b_{1a}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\
&\quad \left. \mathbf{e}^{-\left((-\ln(c_{2a}))^Z + (-\ln(c_{1a}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left((-\ln(d_{2a}))^Z + (-\ln(d_{1a}))^Z \right)^{\frac{1}{Z}}} \right), \\
&\quad \left(\sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-w_{2b}^3))^Z + (-\ln(1-w_{1b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-x_{2b}^3))^Z + (-\ln(1-x_{1b}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\
&\quad \left. \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-y_{2b}^3))^Z + (-\ln(1-y_{1b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-z_{2b}^3))^Z + (-\ln(1-z_{1b}^3))^Z \right)^{\frac{1}{Z}}}}} \right) \rangle \\
&= T_{p_2} \otimes T_{p_1}
\end{aligned}$$

$$(iii) \text{ Let } T_p = \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-a_{1a}^3))^Z + (-\ln(1-a_{2a}^3))^Z \right)^{\frac{1}{Z}}}}. \text{ Then,}$$

$$\ln(1 - T_p) = -\left(\sqrt[3]{(-\ln(1 - a_{1a}^3))^Z + (-\ln(1 - a_{2a}^3))^Z} \right)^{\frac{1}{Z}}, \text{ Using this result, we obtain}$$

$$\zeta(T_{p_1} \oplus T_{p_2})$$

$$\begin{aligned}
&= \zeta \left\langle \left(\sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-a_{1a}^3))^Z + (-\ln(1-a_{2a}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-b_{1a}^3))^Z + (-\ln(1-b_{2a}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \right. \\
&\quad \left. \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-c_{1a}^3))^Z + (-\ln(1-c_{2a}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-d_{1a}^3))^Z + (-\ln(1-d_{2a}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\
&\quad \left(\mathbf{e}^{-\left((-\ln(w_{1b}))^Z + (-\ln(w_{2b}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left((-\ln(x_{1b}))^Z + (-\ln(x_{2b}))^Z \right)^{\frac{1}{Z}}}, \right. \\
&\quad \left. \mathbf{e}^{-\left((-\ln(y_{1b}))^Z + (-\ln(y_{2b}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left((-\ln(z_{1b}))^Z + (-\ln(z_{2b}))^Z \right)^{\frac{1}{Z}}} \right) \rangle
\end{aligned}$$

$$\begin{aligned}
&= \left\langle \left(\sqrt[3]{1 - \mathbf{e}^{-\left(\zeta \left((-\ln(1-a_{1a}^3))^Z + (-\ln(1-a_{2a}^3))^Z \right)\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta \left((-\ln(1-b_{1a}^3))^Z + (-\ln(1-b_{2a}^3))^Z \right)\right)}}, \right. \\
&\quad \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta \left((-\ln(1-c_{1a}^3))^Z + (-\ln(1-c_{2a}^3))^Z \right)\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta \left((-\ln(1-d_{1a}^3))^Z + (-\ln(1-d_{2a}^3))^Z \right)\right)}}, \\
&\quad \left(\mathbf{e}^{-\left(\zeta \left((-\ln(w_{1b}))^Z + (-\ln(w_{2b}))^Z \right)\right)}}, \mathbf{e}^{-\left(\zeta \left((-\ln(x_{1b}))^Z + (-\ln(x_{2b}))^Z \right)\right)}}, \right. \\
&\quad \left. \mathbf{e}^{-\left(\zeta \left((-\ln(y_{1b}))^Z + (-\ln(y_{2b}))^Z \right)\right)}}, \mathbf{e}^{-\left(\zeta \left((-\ln(z_{1b}))^Z + (-\ln(z_{2b}))^Z \right)\right)} \right) \right\rangle \\
&= \left\langle \left(\sqrt[3]{1 - \mathbf{e}^{-\left(\zeta(-\ln(1-a_{1a}^3))^Z\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta(-\ln(1-b_{1a}^3))^Z\right)}}, \right. \\
&\quad \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta(-\ln(1-c_{1a}^3))^Z\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta(-\ln(1-d_{1a}^3))^Z\right)}}, \\
&\quad \left(\mathbf{e}^{-\left(\zeta(-\ln(w_{1b}))^Z\right)}}, \mathbf{e}^{-\left(\zeta(-\ln(x_{1b}))^Z\right)}}, \mathbf{e}^{-\left(\zeta(-\ln(y_{1b}))^Z\right)}}, \mathbf{e}^{-\left(\zeta(-\ln(z_{1b}))^Z\right)} \right) \right\rangle \oplus \\
&\quad \left\langle \left(\sqrt[3]{1 - \mathbf{e}^{-\left(\zeta(-\ln(1-a_{2a}^3))^Z\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta(-\ln(1-b_{2a}^3))^Z\right)}}, \right. \\
&\quad \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta(-\ln(1-c_{2a}^3))^Z\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta(-\ln(1-d_{2a}^3))^Z\right)}}, \\
&\quad \left(\mathbf{e}^{-\left(\zeta(-\ln(w_{2b}))^Z\right)}}, \mathbf{e}^{-\left(\zeta(-\ln(x_{2b}))^Z\right)}}, \mathbf{e}^{-\left(\zeta(-\ln(y_{2b}))^Z\right)}}, \mathbf{e}^{-\left(\zeta(-\ln(z_{2b}))^Z\right)} \right) \right\rangle \\
&= \zeta T_{p1} \oplus \zeta T_{p2}
\end{aligned}$$

(iv) $\zeta_1 T_p \oplus \zeta_2 T_p$

$$\begin{aligned}
&= \left\langle \left(\sqrt[3]{1 - \mathbf{e}^{-\left(\zeta_1(-\ln(1-a_a^3))^Z\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta_1(-\ln(1-b_a^3))^Z\right)}}, \right. \\
&\quad \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta_1(-\ln(1-c_a^3))^Z\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta_1(-\ln(1-d_a^3))^Z\right)}}, \\
&\quad \left(\mathbf{e}^{-\left(\zeta_1(-\ln(w_b))^Z\right)}}, \mathbf{e}^{-\left(\zeta_1(-\ln(x_b))^Z\right)}}, \mathbf{e}^{-\left(\zeta_1(-\ln(y_b))^Z\right)}}, \mathbf{e}^{-\left(\zeta_1(-\ln(z_b))^Z\right)} \right) \right\rangle \\
&\oplus \left\langle \left(\sqrt[3]{1 - \mathbf{e}^{-\left(\zeta_2(-\ln(1-a_a^3))^Z\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta_2(-\ln(1-b_a^3))^Z\right)}}, \right. \\
&\quad \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta_2(-\ln(1-c_a^3))^Z\right)}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\zeta_2(-\ln(1-d_a^3))^Z\right)}}, \\
&\quad \left(\mathbf{e}^{-\left(\zeta_2(-\ln(w_b))^Z\right)}}, \mathbf{e}^{-\left(\zeta_2(-\ln(x_b))^Z\right)}}, \mathbf{e}^{-\left(\zeta_2(-\ln(y_b))^Z\right)}}, \mathbf{e}^{-\left(\zeta_2(-\ln(z_b))^Z\right)} \right) \right\rangle
\end{aligned}$$

$$\begin{aligned}
& \left(e^{-\left(\zeta_2(-\ln(w_b))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_2(-\ln(x_b))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_2(-\ln(y_b))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_2(-\ln(z_b))^Z\right)^{\frac{1}{Z}}} \right) \rangle \\
& = \left\langle \left(\sqrt[3]{1 - e^{-\left((\zeta_1 + \zeta_2)(-\ln(1-a_a^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((\zeta_1 + \zeta_2)(-\ln(1-b_a^3))^Z\right)^{\frac{1}{Z}}}}, \right. \right. \\
& \quad \left. \sqrt[3]{1 - e^{-\left((\zeta_1 + \zeta_2)(-\ln(1-c_a^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((\zeta_1 + \zeta_2)(-\ln(1-d_a^3))^Z\right)^{\frac{1}{Z}}}} \right), \\
& \quad \left(e^{-\left((\zeta_1 + \zeta_2)(-\ln(w_b))^Z\right)^{\frac{1}{Z}}}, e^{-\left((\zeta_1 + \zeta_2)(-\ln(x_b))^Z\right)^{\frac{1}{Z}}}, e^{-\left((\zeta_1 + \zeta_2)(-\ln(y_b))^Z\right)^{\frac{1}{Z}}}, e^{-\left((\zeta_1 + \zeta_2)(-\ln(z_b))^Z\right)^{\frac{1}{Z}}} \right) \rangle \\
& = (\zeta_1 + \zeta_2) T_p \\
\text{(v)} \quad (T_{p1} \otimes T_{p2})^\zeta &= \\
& \zeta \left\langle \left(e^{-\left((-\ln(a_{1a}))^Z + (-\ln(a_{2a}))^Z\right)^{\frac{1}{Z}}}, e^{-\left((-\ln(b_{1a}))^Z + (-\ln(b_{2a}))^Z\right)^{\frac{1}{Z}}}, \right. \right. \\
& \quad e^{-\left((-\ln(c_{1a}))^Z + (-\ln(c_{2a}))^Z\right)^{\frac{1}{Z}}}, e^{-\left((-\ln(d_{1a}))^Z + (-\ln(d_{2a}))^Z\right)^{\frac{1}{Z}}} \right), \\
& \quad \left(\sqrt[3]{1 - e^{-\left((-\ln(1-w_{1b}^3))^Z + (-\ln(1-w_{2b}^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((-\ln(1-x_{1b}^3))^Z + (-\ln(1-x_{2b}^3))^Z\right)^{\frac{1}{Z}}}}, \right. \\
& \quad \left. \sqrt[3]{1 - e^{-\left((-\ln(1-y_{1b}^3))^Z + (-\ln(1-y_{2b}^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((-\ln(1-z_{1b}^3))^Z + (-\ln(1-z_{2b}^3))^Z\right)^{\frac{1}{Z}}}} \right) \rangle \\
& = \left\langle \left(e^{-\left(\zeta \left((-\ln(a_{1a}))^Z + (-\ln(a_{2a}))^Z \right)\right)^{\frac{1}{Z}}}, e^{-\left(\zeta \left((-\ln(b_{1a}))^Z + (-\ln(b_{2a}))^Z \right)\right)^{\frac{1}{Z}}}, \right. \right. \\
& \quad e^{-\left(\zeta \left((-\ln(c_{1a}))^Z + (-\ln(c_{2a}))^Z \right)\right)^{\frac{1}{Z}}}, e^{-\left(\zeta \left((-\ln(d_{1a}))^Z + (-\ln(d_{2a}))^Z \right)\right)^{\frac{1}{Z}}} \right), \\
& \quad \left(\sqrt[3]{1 - e^{-\left(\zeta \left((-\ln(1-w_{1b}^3))^Z + (-\ln(1-w_{2b}^3))^Z \right)\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta \left((-\ln(1-x_{1b}^3))^Z + (-\ln(1-x_{2b}^3))^Z \right)\right)^{\frac{1}{Z}}}}, \right. \\
& \quad \left. \sqrt[3]{1 - e^{-\left(\zeta \left((-\ln(1-y_{1b}^3))^Z + (-\ln(1-y_{2b}^3))^Z \right)\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta \left((-\ln(1-z_{1b}^3))^Z + (-\ln(1-z_{2b}^3))^Z \right)\right)^{\frac{1}{Z}}}} \right) \rangle \\
& = \left\langle \left(e^{-\left(\zeta(-\ln(a_{1a}))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(b_{1a}))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(c_{1a}))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(d_{1a}))^Z\right)^{\frac{1}{Z}}}, \right. \right. \\
& \quad \left(\sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-w_{1b}^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-x_{1b}^3))^Z\right)^{\frac{1}{Z}}}}, \right. \\
& \quad \left. \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-y_{1b}^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-z_{1b}^3))^Z\right)^{\frac{1}{Z}}}} \right) \rangle
\end{aligned}$$

$$\begin{aligned}
& \otimes \left\langle \left(e^{-\left(\zeta(-\ln(a_{2a}))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(b_{2a}))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(c_{2a}))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta(-\ln(d_{2a}))^Z\right)^{\frac{1}{Z}}} \right), \right. \\
& \left(\sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-w_{2b}^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-x_{2b}^3))^Z\right)^{\frac{1}{Z}}}}, \right. \\
& \left. \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-y_{2b}^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta(-\ln(1-z_{2b}^3))^Z\right)^{\frac{1}{Z}}}} \right) \rangle \\
& = (T_{p_1}^\zeta \otimes T_{p_2}^\zeta) \\
& \text{(vi) } T_p^{\zeta_1} \otimes T_p^{\zeta_2} \\
& = \left\langle \left(e^{-\left(\zeta_1(-\ln(a_a))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_1(-\ln(b_a))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_1(-\ln(c_a))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_1(-\ln(d_a))^Z\right)^{\frac{1}{Z}}} \right), \right. \\
& \left(\sqrt[3]{1 - e^{-\left(\zeta_1(-\ln(1-w_b^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta_1(-\ln(1-x_b^3))^Z\right)^{\frac{1}{Z}}}}, \right. \\
& \left. \sqrt[3]{1 - e^{-\left(\zeta_1(-\ln(1-y_b^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta_1(-\ln(1-z_b^3))^Z\right)^{\frac{1}{Z}}}} \right) \rangle \\
& \otimes \left\langle \left(e^{-\left(\zeta_2(-\ln(a_a))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_2(-\ln(b_a))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_2(-\ln(c_a))^Z\right)^{\frac{1}{Z}}}, e^{-\left(\zeta_2(-\ln(d_a))^Z\right)^{\frac{1}{Z}}} \right), \right. \\
& \left(\sqrt[3]{1 - e^{-\left(\zeta_2(-\ln(1-w_b^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta_2(-\ln(1-x_b^3))^Z\right)^{\frac{1}{Z}}}}, \right. \\
& \left. \sqrt[3]{1 - e^{-\left(\zeta_2(-\ln(1-y_b^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\zeta_2(-\ln(1-z_b^3))^Z\right)^{\frac{1}{Z}}}} \right) \rangle \\
& = \left\langle \left(e^{-\left((\zeta_1+\zeta_2)(-\ln(a_a))^Z\right)^{\frac{1}{Z}}}, e^{-\left((\zeta_1+\zeta_2)(-\ln(b_a))^Z\right)^{\frac{1}{Z}}}, \right. \\
& \left. e^{-\left((\zeta_1+\zeta_2)(-\ln(c_a))^Z\right)^{\frac{1}{Z}}}, e^{-\left((\zeta_1+\zeta_2)(-\ln(d_a))^Z\right)^{\frac{1}{Z}}} \right), \\
& \left(\sqrt[3]{1 - e^{-\left((\zeta_1+\zeta_2)(-\ln(1-w_b^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((\zeta_1+\zeta_2)(-\ln(1-x_b^3))^Z\right)^{\frac{1}{Z}}}}, \right. \\
& \left. \sqrt[3]{1 - e^{-\left((\zeta_1+\zeta_2)(-\ln(1-y_b^3))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left((\zeta_1+\zeta_2)(-\ln(1-z_b^3))^Z\right)^{\frac{1}{Z}}}} \right) \rangle \\
& = T_p^{\zeta_1+\zeta_2}
\end{aligned}$$

□

We present several geometric $\mathcal{AO}$ s for TpVFF numbers and examine their characteristics in the next section.

4. Trapezoidal-valued Fermatean fuzzy Aczel–Alsina geometric aggregation operators

In this section, we introduce several $\mathcal{AA}$ -based geometric aggregation operators for TpVFF numbers, namely, $TpVFFAAWG$, $TpVFFAAOWG$, and $TpVFFAAHG$, and establish their aggre-

gation properties through the corresponding theorems.

Definition 9. For $\mathcal{V} = 1, 2, \dots, \mathcal{Q}$, let $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$ be a collection of TpVFF numbers, and let $\check{w}_{\mathcal{V}} = (\check{w}_1, \check{w}_2, \dots, \check{w}_{\mathcal{Q}})^T$ be the corresponding weight vector, where $\check{w}_{\mathcal{V}} \in [0, 1]$ and $\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} = 1$. Then, the $TpVFFAAWG$ operator is defined as follows:

$$TpVFFAAWG(T_{p_1}, T_{p_2}, \dots, T_{p_{\mathcal{Q}}}) = \otimes_{\mathcal{V}=1}^{\mathcal{Q}} (T_{p_{\mathcal{V}}})^{\check{w}_{\mathcal{V}}} = (T_{p_1})^{\check{w}_1} \otimes (T_{p_2})^{\check{w}_2} \otimes \dots \otimes (T_{p_{\mathcal{Q}}})^{\check{w}_{\mathcal{Q}}}$$

The $TpVFFAAWG$ operator is defined in Definition 9, and Theorems 2 to 5 establish its mathematical properties.

Theorem 2. Let $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$, ($\mathcal{V} = 1, 2, \dots, \mathcal{Q}$), be a collection of TpVFF numbers. Then, the aggregated value obtained using the $TpVFFAAWG$ operator is also a TpVFF number.

$$\begin{aligned} TpVFFAAWG(T_{p_1}, T_{p_2}, \dots, T_{p_{\mathcal{Q}}}) &= \otimes_{\mathcal{V}=1}^{\mathcal{Q}} (T_{p_{\mathcal{V}}})^{\check{w}_{\mathcal{V}}} \\ &= \left\langle \left(e^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} (-\ln(a_{\mathcal{V}\hat{a}}))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, e^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} (-\ln(b_{\mathcal{V}\hat{a}}))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \right. \\ &\quad \left. \left(e^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} (-\ln(c_{\mathcal{V}\hat{a}}))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, e^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} (-\ln(d_{\mathcal{V}\hat{a}}))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \right. \right. \\ &\quad \left. \left(\sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} (-\ln(1-w_{\mathcal{V}\hat{b}}^3))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} (-\ln(1-x_{\mathcal{V}\hat{b}}^3))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \right. \right. \\ &\quad \left. \left. \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} (-\ln(1-y_{\mathcal{V}\hat{b}}^3))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} (-\ln(1-z_{\mathcal{V}\hat{b}}^3))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}} \right) \right\rangle \end{aligned} \quad (1)$$

where ($\mathcal{V} = 1, 2, \dots, \mathcal{Q}$), $\check{w}_{\mathcal{V}} \in [0, 1]$ and $\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}} = 1$, with $\check{w}_{\mathcal{V}} = (\check{w}_1, \check{w}_2, \dots, \check{w}_{\mathcal{Q}})^T$ denoting the weight vector of $T_{p_{\mathcal{V}}}$.

Proof. The theorem is established using mathematical induction. For the base case $\mathcal{Q} = 2$, the result follows directly by applying the $\mathcal{AA}$ operation of TpVFF numbers. Thus,

$$\begin{aligned} TpVFFAAWG(T_{p_1}, T_{p_2}) &= \otimes_{\mathcal{V}=1}^2 (T_{p_{\mathcal{V}}})^{\check{w}_{\mathcal{V}}} = (T_{p_1})^{\check{w}_1} \otimes (T_{p_2})^{\check{w}_2} \\ &= \left\langle \left(e^{-\left(\check{w}_1 (-\ln(a_{1\hat{a}}))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, e^{-\left(\check{w}_1 (-\ln(b_{1\hat{a}}))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, e^{-\left(\check{w}_1 (-\ln(c_{1\hat{a}}))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, e^{-\left(\check{w}_1 (-\ln(d_{1\hat{a}}))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \right. \right. \\ &\quad \left. \left(\sqrt[3]{1 - e^{-\left(\check{w}_1 (-\ln(1-w_{1\hat{b}}^3))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \sqrt[3]{1 - e^{-\left(\check{w}_1 (-\ln(1-x_{1\hat{b}}^3))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \right. \right. \\ &\quad \left. \left. \sqrt[3]{1 - e^{-\left(\check{w}_1 (-\ln(1-y_{1\hat{b}}^3))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}, \sqrt[3]{1 - e^{-\left(\check{w}_1 (-\ln(1-z_{1\hat{b}}^3))^{\mathcal{Z}} \right)^{\frac{1}{\mathcal{Z}}}}} \right) \right\rangle \end{aligned}$$

$$\begin{aligned}
& \otimes \left\langle \left(e^{-\left(\check{w}_2(-\ln(a_{2a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\check{w}_2(-\ln(b_{2a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\check{w}_2(-\ln(c_{2a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\check{w}_2(-\ln(d_{2a}))^Z \right)^{\frac{1}{Z}}} \right), \right. \\
& \left(\sqrt[3]{1 - e^{-\left(\check{w}_2(-\ln(1-w_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\check{w}_2(-\ln(1-x_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\
& \left. \sqrt[3]{1 - e^{-\left(\check{w}_2(-\ln(1-y_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\check{w}_2(-\ln(1-z_{2b}^3))^Z \right)^{\frac{1}{Z}}}} \right) \rangle \\
& = \left\langle \left(e^{-\left(\check{w}_1(-\ln(a_{1a}))^Z + \check{w}_2(-\ln(a_{2a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\check{w}_1(-\ln(b_{1a}))^Z + \check{w}_2(-\ln(b_{2a}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\
& e^{-\left(\check{w}_1(-\ln(c_{1a}))^Z + \check{w}_2(-\ln(c_{2a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\check{w}_1(-\ln(d_{1a}))^Z + \check{w}_2(-\ln(d_{2a}))^Z \right)^{\frac{1}{Z}}} \right), \\
& \left(\sqrt[3]{1 - e^{-\left(\check{w}_1(-\ln(1-w_{1b}^3))^Z + \check{w}_2(-\ln(1-w_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\check{w}_1(-\ln(1-x_{1b}^3))^Z + \check{w}_2(-\ln(1-x_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\
& \left. \sqrt[3]{1 - e^{-\left(\check{w}_1(-\ln(1-y_{1b}^3))^Z + \check{w}_2(-\ln(1-y_{2b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\check{w}_1(-\ln(1-z_{1b}^3))^Z + \check{w}_2(-\ln(1-z_{2b}^3))^Z \right)^{\frac{1}{Z}}}} \right) \rangle \\
& = \left\langle \left(e^{-\left(\sum_{\nu=1}^2 \check{w}_{\nu}(-\ln(a_{\nu a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\sum_{\nu=1}^2 \check{w}_{\nu}(-\ln(b_{\nu a}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\
& \left\langle \left(e^{-\left(\sum_{\nu=1}^2 \check{w}_{\nu}(-\ln(c_{\nu a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\sum_{\nu=1}^2 \check{w}_{\nu}(-\ln(d_{\nu a}))^Z \right)^{\frac{1}{Z}}}, \right. \\
& \left(\sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^2 \check{w}_{\nu}(-\ln(1-w_{\nu b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^2 \check{w}_{\nu}(-\ln(1-x_{\nu b}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\
& \left. \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^2 \check{w}_{\nu}(-\ln(1-y_{\nu b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^2 \check{w}_{\nu}(-\ln(1-z_{\nu b}^3))^Z \right)^{\frac{1}{Z}}}} \right) \rangle
\end{aligned}$$

Thus, the result holds for $\mathcal{Q} = 2$. Now, assume that it holds for $\mathcal{Q} = \varepsilon$. Then,

$$\begin{aligned}
& TpVFFAAG(T_{p_1}, T_{p_2}, \dots, T_{p_\varepsilon}) = \otimes_{\nu=1}^\varepsilon (T_{p_\nu})^{\check{w}_\nu} \\
& = \left\langle \left(e^{-\left(\sum_{\nu=1}^\varepsilon \check{w}_\nu(-\ln(a_{\nu a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\sum_{\nu=1}^\varepsilon \check{w}_\nu(-\ln(b_{\nu a}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\
& \left\langle \left(e^{-\left(\sum_{\nu=1}^\varepsilon \check{w}_\nu(-\ln(c_{\nu a}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\sum_{\nu=1}^\varepsilon \check{w}_\nu(-\ln(d_{\nu a}))^Z \right)^{\frac{1}{Z}}}, \right. \\
& \left(\sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^\varepsilon \check{w}_\nu(-\ln(1-w_{\nu b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^\varepsilon \check{w}_\nu(-\ln(1-x_{\nu b}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \\
& \left. \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^\varepsilon \check{w}_\nu(-\ln(1-y_{\nu b}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^\varepsilon \check{w}_\nu(-\ln(1-z_{\nu b}^3))^Z \right)^{\frac{1}{Z}}}} \right) \rangle
\end{aligned}$$

We now prove the result for $\mathcal{Q} = \varepsilon + 1$.

$$TpVFFAAG(T_{p_1}, T_{p_2}, \dots, T_{p_\varepsilon}, T_{p_{\varepsilon+1}}) = \otimes_{\nu=1}^{\varepsilon+1} (T_{p_\nu})^{\check{w}_\nu} = \otimes_{\nu=1}^\varepsilon (T_{p_\nu})^{\check{w}_\nu} \otimes (T_{p_{\varepsilon+1}})^{\check{w}_{\varepsilon+1}}$$

$$\begin{aligned}
&= \left\langle \left(e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon} \check{w}_{\mathcal{V}} (-\ln(a_{\mathcal{V}\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon} \check{w}_{\mathcal{V}} (-\ln(b_{\mathcal{V}\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, \right. \right. \\
&\quad \left\langle \left(e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon} \check{w}_{\mathcal{V}} (-\ln(c_{\mathcal{V}\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon} \check{w}_{\mathcal{V}} (-\ln(d_{\mathcal{V}\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, \right. \right. \\
&\quad \left(\sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon} \check{w}_{\mathcal{V}} (-\ln(1-w_{\mathcal{V}\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon} \check{w}_{\mathcal{V}} (-\ln(1-x_{\mathcal{V}\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \right. \\
&\quad \left. \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon} \check{w}_{\mathcal{V}} (-\ln(1-y_{\mathcal{V}\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon} \check{w}_{\mathcal{V}} (-\ln(1-z_{\mathcal{V}\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}} \right) \rangle \\
&\quad \otimes \left\langle \left(e^{-\left(\check{w}_{\varepsilon+1} (-\ln(a_{\varepsilon+1\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\check{w}_{\varepsilon+1} (-\ln(b_{\varepsilon+1\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\check{w}_{\varepsilon+1} (-\ln(c_{\varepsilon+1\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\check{w}_{\varepsilon+1} (-\ln(d_{\varepsilon+1\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, \right. \right. \\
&\quad \left(\sqrt[3]{1 - e^{-\left(\check{w}_{\varepsilon+1} (-\ln(1-w_{\varepsilon+1\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \sqrt[3]{1 - e^{-\left(\check{w}_{\varepsilon+1} (-\ln(1-x_{\varepsilon+1\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \right. \\
&\quad \left. \sqrt[3]{1 - e^{-\left(\check{w}_{\varepsilon+1} (-\ln(1-y_{\varepsilon+1\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \sqrt[3]{1 - e^{-\left(\check{w}_{\varepsilon+1} (-\ln(1-z_{\varepsilon+1\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}} \right) \rangle \\
&= \left\langle \left(e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon+1} \check{w}_{\mathcal{V}} (-\ln(a_{\mathcal{V}\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon+1} \check{w}_{\mathcal{V}} (-\ln(b_{\mathcal{V}\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, \right. \right. \\
&\quad \left\langle \left(e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon+1} \check{w}_{\mathcal{V}} (-\ln(c_{\mathcal{V}\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon+1} \check{w}_{\mathcal{V}} (-\ln(d_{\mathcal{V}\hat{a}}))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}, \right. \right. \\
&\quad \left(\sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon+1} \check{w}_{\mathcal{V}} (-\ln(1-w_{\mathcal{V}\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon+1} \check{w}_{\mathcal{V}} (-\ln(1-x_{\mathcal{V}\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \right. \\
&\quad \left. \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon+1} \check{w}_{\mathcal{V}} (-\ln(1-y_{\mathcal{V}\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^{\varepsilon+1} \check{w}_{\mathcal{V}} (-\ln(1-z_{\mathcal{V}\hat{b}}^3))^{\mathbb{Z}} \right)^{\frac{1}{\mathbb{Z}}}}} \right) \rangle
\end{aligned}$$

Hence, Equation (1) holds for $\mathcal{Q} = \varepsilon + 1$. Therefore, by mathematical induction, Equation 1 holds for any positive integer $\mathcal{Q}$.

Since all input values are TpVFF numbers, the proposed aggregation operator preserves the range $[0, 1]$. By the Fermatean condition and the monotonicity of the aggregation operators, the aggregated result satisfies the Fermatean constraint and preserves the required trapezoidal ordering. Therefore, the aggregated value is also a *TpVFFN*, which proves the closure property. $\square$

The aggregation operator *TpVFFAAWG* is defined as follows. The aggregation properties, including idempotency, boundedness, and monotonicity, are established through Theorems 3, 4, and 5.

Theorem 3. Suppose that $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$ for all $(\mathcal{V} = 1, 2, \dots, \mathcal{Q})$. Then, $TpVFFAAWG(T_{p_1}, T_{p_2}, \dots, T_{p_{\mathcal{Q}}}) = T_p$ (Idempotency)

Proof. Since, $T_{p_{\mathcal{V}}} = T_p$ for $(\mathcal{V} = 1, 2, \dots, \mathcal{Q})$, that is, $a_{\mathcal{V}\hat{a}} = a_{\hat{a}}$, $b_{\mathcal{V}\hat{a}} = b_{\hat{a}}$, $c_{\mathcal{V}\hat{a}} = c_{\hat{a}}$, $d_{\mathcal{V}\hat{a}} = d_{\hat{a}}$, and $w_{\mathcal{V}\hat{b}} = w_{\hat{b}}$, $x_{\mathcal{V}\hat{b}} = x_{\hat{b}}$, $y_{\mathcal{V}\hat{b}} = y_{\hat{b}}$, $z_{\mathcal{V}\hat{b}} = z_{\hat{b}}$, we have

$$\begin{aligned}
TpVFFAAG(T_{p_1}, T_{p_2}, \dots, T_{p_Q}) &= \otimes_{\mathcal{V}=1}^Q (T_{p_{\mathcal{V}}})^{\tilde{w}_{\mathcal{V}}} \\
&= \left\langle \left(\mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^Q \tilde{w}_{\mathcal{V}} (-\ln(a_{\mathcal{V}\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^Q \tilde{w}_{\mathcal{V}} (-\ln(b_{\mathcal{V}\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\
&\quad \left. \left(\mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^Q \tilde{w}_{\mathcal{V}} (-\ln(c_{\mathcal{V}\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^Q \tilde{w}_{\mathcal{V}} (-\ln(d_{\mathcal{V}\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\
&\quad \left. \left(\sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^Q \tilde{w}_{\mathcal{V}} (-\ln(1-w_{\mathcal{V}\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^Q \tilde{w}_{\mathcal{V}} (-\ln(1-x_{\mathcal{V}\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \right. \\
&\quad \left. \left. \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^Q \tilde{w}_{\mathcal{V}} (-\ln(1-y_{\mathcal{V}\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^Q \tilde{w}_{\mathcal{V}} (-\ln(1-z_{\mathcal{V}\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}} \right) \right\rangle \\
&= \left\langle \left(\mathbf{e}^{-\left((-\ln(a_{\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left((-\ln(b_{\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \left(\mathbf{e}^{-\left((-\ln(c_{\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left((-\ln(d_{\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \right. \\
&\quad \left. \left(\sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-w_{\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-x_{\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \right. \\
&\quad \left. \left. \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-y_{\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left((-\ln(1-z_{\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}} \right) \right\rangle \\
&= \left\langle \left(\mathbf{e}^{\ln(a_{\hat{a}})}, \mathbf{e}^{\ln(b_{\hat{a}})}, \mathbf{e}^{\ln(c_{\hat{a}})}, \mathbf{e}^{\ln(d_{\hat{a}})}, \right. \right. \\
&\quad \left. \left(\sqrt[3]{1 - \mathbf{e}^{-\ln(1-w_{\hat{b}}^3)}}, \sqrt[3]{1 - \mathbf{e}^{-\ln(1-x_{\hat{b}}^3)}}, \sqrt[3]{1 - \mathbf{e}^{-\ln(1-y_{\hat{b}}^3)}}, \sqrt[3]{1 - \mathbf{e}^{-\ln(1-z_{\hat{b}}^3)}} \right) \right\rangle \\
&= \langle (a_{\hat{a}}, b_{\hat{a}}, c_{\hat{a}}, d_{\hat{a}}), (w_{\hat{b}}, x_{\hat{b}}, y_{\hat{b}}, z_{\hat{b}}) \rangle = T_p
\end{aligned}$$

□

Theorem 4. Let $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$, ($\mathcal{V} = 1, 2, \dots, Q$), be a collection of TpVFF numbers, and let $T_{p_{\mathcal{V}}}^- = \text{Min}\{T_{p_1}, T_{p_2}, \dots, T_{p_Q}\}$ and $T_{p_{\mathcal{V}}}^+ = \text{Max}\{T_{p_1}, T_{p_2}, \dots, T_{p_Q}\}$ denote their minimum and maximum values, respectively. Then, $T_{p_{\mathcal{V}}}^- \leq TpVFFAAG(T_{p_1}, T_{p_2}, \dots, T_{p_Q}) \leq T_{p_{\mathcal{V}}}^+$. (Boundedness)

Proof. Let $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$, ($\mathcal{V} = 1, 2, \dots, Q$) be a collection of TpVFF numbers and let

$$\begin{aligned}
T_{p_{\mathcal{V}}}^- &= \text{Min}\{T_{p_1}, T_{p_2}, \dots, T_{p_Q}\} = \langle (a_{\hat{a}}^-, b_{\hat{a}}^-, c_{\hat{a}}^-, d_{\hat{a}}^-), (w_{\hat{b}}^-, x_{\hat{b}}^-, y_{\hat{b}}^-, z_{\hat{b}}^-) \rangle \text{ and} \\
T_{p_{\mathcal{V}}}^+ &= \text{Max}\{T_{p_1}, T_{p_2}, \dots, T_{p_Q}\} = \langle (a_{\hat{a}}^+, b_{\hat{a}}^+, c_{\hat{a}}^+, d_{\hat{a}}^+), (w_{\hat{b}}^+, x_{\hat{b}}^+, y_{\hat{b}}^+, z_{\hat{b}}^+) \rangle,
\end{aligned}$$

where

$$\begin{aligned}
a_{\hat{a}}^- &= \text{Min}_{\mathcal{V}}\{a_{\mathcal{V}\hat{a}}\}, & b_{\hat{a}}^- &= \text{Min}_{\mathcal{V}}\{b_{\mathcal{V}\hat{a}}\}, & c_{\hat{a}}^- &= \text{Min}_{\mathcal{V}}\{c_{\mathcal{V}\hat{a}}\}, & d_{\hat{a}}^- &= \text{Min}_{\mathcal{V}}\{d_{\mathcal{V}\hat{a}}\}, \\
w_{\hat{b}}^- &= \text{Max}_{\mathcal{V}}\{w_{\mathcal{V}\hat{b}}\}, & x_{\hat{b}}^- &= \text{Max}_{\mathcal{V}}\{x_{\mathcal{V}\hat{b}}\}, & y_{\hat{b}}^- &= \text{Max}_{\mathcal{V}}\{y_{\mathcal{V}\hat{b}}\}, & z_{\hat{b}}^- &= \text{Max}_{\mathcal{V}}\{z_{\mathcal{V}\hat{b}}\}, \text{ and} \\
a_{\hat{a}}^+ &= \text{Max}_{\mathcal{V}}\{a_{\mathcal{V}\hat{a}}\}, & b_{\hat{a}}^+ &= \text{Max}_{\mathcal{V}}\{b_{\mathcal{V}\hat{a}}\}, & c_{\hat{a}}^+ &= \text{Max}_{\mathcal{V}}\{c_{\mathcal{V}\hat{a}}\}, & d_{\hat{a}}^+ &= \text{Max}_{\mathcal{V}}\{d_{\mathcal{V}\hat{a}}\}, \\
w_{\hat{b}}^+ &= \text{Min}_{\mathcal{V}}\{w_{\mathcal{V}\hat{b}}\}, & x_{\hat{b}}^+ &= \text{Min}_{\mathcal{V}}\{x_{\mathcal{V}\hat{b}}\}, & y_{\hat{b}}^+ &= \text{Min}_{\mathcal{V}}\{y_{\mathcal{V}\hat{b}}\}, & z_{\hat{b}}^+ &= \text{Min}_{\mathcal{V}}\{z_{\mathcal{V}\hat{b}}\},
\end{aligned}$$

$$\begin{aligned}
e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(a_{\hat{a}}^-)^Z)\right)^{\frac{1}{Z}}} &\leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(a_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} \leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(a_{\hat{a}}^+)^Z)\right)^{\frac{1}{Z}}} \\
e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(b_{\hat{a}}^-)^Z)\right)^{\frac{1}{Z}}} &\leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(b_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} \leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(b_{\hat{a}}^+)^Z)\right)^{\frac{1}{Z}}} \\
e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(c_{\hat{a}}^-)^Z)\right)^{\frac{1}{Z}}} &\leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(c_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} \leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(c_{\hat{a}}^+)^Z)\right)^{\frac{1}{Z}}} \\
e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(d_{\hat{a}}^-)^Z)\right)^{\frac{1}{Z}}} &\leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(d_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} \leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(d_{\hat{a}}^+)^Z)\right)^{\frac{1}{Z}}}
\end{aligned}$$

and

$$\begin{aligned}
\sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-w_{\hat{b}}^{3+})^Z)\right)^{\frac{1}{Z}}}} &\leq \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-w_{\mathcal{V}\hat{b}}^3)^Z)\right)^{\frac{1}{Z}}}} \\
&\leq \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-w_{\hat{b}}^{3-})^Z)\right)^{\frac{1}{Z}}}} \\
\sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-x_{\hat{b}}^{3+})^Z)\right)^{\frac{1}{Z}}}} &\leq \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-x_{\mathcal{V}\hat{b}}^3)^Z)\right)^{\frac{1}{Z}}}} \\
&\leq \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-x_{\hat{b}}^{3-})^Z)\right)^{\frac{1}{Z}}}} \\
\sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-y_{\hat{b}}^{3+})^Z)\right)^{\frac{1}{Z}}}} &\leq \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-y_{\mathcal{V}\hat{b}}^3)^Z)\right)^{\frac{1}{Z}}}} \\
&\leq \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-y_{\hat{b}}^{3-})^Z)\right)^{\frac{1}{Z}}}} \\
\sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-z_{\hat{b}}^{3+})^Z)\right)^{\frac{1}{Z}}}} &\leq \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-z_{\mathcal{V}\hat{b}}^3)^Z)\right)^{\frac{1}{Z}}}} \\
&\leq \sqrt[3]{1 - e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(1-z_{\hat{b}}^{3-})^Z)\right)^{\frac{1}{Z}}}}
\end{aligned}$$

Therefore,

$$T_{p_{\mathcal{V}}}^- \leq TpVFFAAWG(T_{p_1}, T_{p_2}, \dots, T_{p_Q}) \leq T_{p_{\mathcal{V}}}^+$$

□

Theorem 5. Let $T_{p_{\mathcal{V}}} = \widehat{T}_{p_{\mathcal{V}}}$, ($\mathcal{V} = 1, 2, \dots, Q$), be two collections of TpVFF numbers. If $T_{p_{\mathcal{V}}} \leq \widehat{T}_{p_{\mathcal{V}}}$ for all $\mathcal{V}$, then $TpVFFAAWG(T_{p_1}, T_{p_2}, \dots, T_{p_Q}) \leq TpVFFAAWG(\widehat{T}_{p_1}, \widehat{T}_{p_2}, \dots, \widehat{T}_{p_Q})$, where $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$ and $\widehat{T}_{p_{\mathcal{V}}} = \langle (\hat{a}_{\mathcal{V}\hat{a}}, \hat{b}_{\mathcal{V}\hat{a}}, \hat{c}_{\mathcal{V}\hat{a}}, \hat{d}_{\mathcal{V}\hat{a}}), (\hat{w}_{\mathcal{V}\hat{b}}, \hat{x}_{\mathcal{V}\hat{b}}, \hat{y}_{\mathcal{V}\hat{b}}, \hat{z}_{\mathcal{V}\hat{b}}) \rangle$ (Monotonicity)

Proof. Since $T_{p_{\mathcal{V}}} = \widehat{T}_{p_{\mathcal{V}}}$, that is $a_{\mathcal{V}\hat{a}} \leq \hat{a}_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}} \leq \hat{b}_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}} \leq \hat{c}_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}} \leq \hat{d}_{\mathcal{V}\hat{a}}$ and $\hat{w}_{\mathcal{V}\hat{b}} \geq w_{\mathcal{V}\hat{b}}, \hat{x}_{\mathcal{V}\hat{b}} \geq x_{\mathcal{V}\hat{b}}, \hat{y}_{\mathcal{V}\hat{b}} \geq y_{\mathcal{V}\hat{b}}, \hat{z}_{\mathcal{V}\hat{b}} \geq z_{\mathcal{V}\hat{b}}$

$$\begin{aligned}
e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(a_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} &\leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(\hat{a}_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} \\
e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(b_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} &\leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(\hat{b}_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} \\
e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(c_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} &\leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(\hat{c}_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} \\
e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(d_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}} &\leq e^{-\left(\sum_{\mathcal{V}=1}^Q \check{w}_{\mathcal{V}}(-\ln(\hat{d}_{\mathcal{V}\hat{a}})^Z)\right)^{\frac{1}{Z}}}
\end{aligned}$$

and

$$\begin{aligned} \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}}(-\ln(1-w_{\mathcal{V}\hat{b}}^3))^Z\right)^{\frac{1}{Z}}}} &\geq \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}}(-\ln(1-\hat{w}_{\mathcal{V}\hat{b}}^3))^Z\right)^{\frac{1}{Z}}}} \\ \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}}(-\ln(1-x_{\mathcal{V}\hat{b}}^3))^Z\right)^{\frac{1}{Z}}}} &\geq \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}}(-\ln(1-\hat{x}_{\mathcal{V}\hat{b}}^3))^Z\right)^{\frac{1}{Z}}}} \\ \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}}(-\ln(1-y_{\mathcal{V}\hat{b}}^3))^Z\right)^{\frac{1}{Z}}}} &\geq \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}}(-\ln(1-\hat{y}_{\mathcal{V}\hat{b}}^3))^Z\right)^{\frac{1}{Z}}}} \\ \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}}(-\ln(1-z_{\mathcal{V}\hat{b}}^3))^Z\right)^{\frac{1}{Z}}}} &\geq \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{w}_{\mathcal{V}}(-\ln(1-\hat{z}_{\mathcal{V}\hat{b}}^3))^Z\right)^{\frac{1}{Z}}}} \end{aligned}$$

By combining all the above conditions, we obtain

$$TpVFFAAWG(T_{p_1}, T_{p_2}, \dots, T_{p_Q}) \leq TpVFFAAWG(\widehat{T}_{p_1}, \widehat{T}_{p_2}, \dots, \widehat{T}_{p_Q}).$$

□

We now introduce the $TpVFFAAOWG$ operator and establish its properties through 6 to 9.

Definition 10. Let $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$, $(\mathcal{V} = 1, 2, \dots, \mathcal{Q})$ be a collection of $TpVFF$ numbers. A mapping of $TpVFFAAOWG$ of dimension $\mathcal{Q}$ is a $TpVFFAAOWG : (T_p^*)^{\mathcal{Q}} \rightarrow (T_p^*)$ with a weight vector $\check{v}_{\mathcal{V}} = (\check{v}_1, \check{v}_2, \dots, \check{v}_{\mathcal{Q}})^T$ of $T_{p_{\mathcal{V}}}$ with $\check{v}_{\mathcal{V}} \in [0, 1]$ and $\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}} = 1$, defined as follows:

$$TpVFFAAOWG(T_{p_1}, T_{p_2}, \dots, T_{p_Q}) = \otimes_{\mathcal{V}=1}^{\mathcal{Q}} (T_{p_{f(\mathcal{V})}})^{\check{v}_{\mathcal{V}}} = (T_{p_{f(1)}})^{\check{v}_1} \otimes (T_{p_{f(2)}})^{\check{v}_2} \otimes \dots \otimes (T_{p_{f(\mathcal{Q})}})^{\check{v}_{\mathcal{Q}}}$$

where $(f(1), f(2), \dots, f(\mathcal{Q}))$ is a permutation of $(1, 2, \dots, \mathcal{Q})$, for which $(T_{p_{f(\mathcal{V}-1)}}) \geq (T_{p_{f(\mathcal{V})}})$ for all $(\mathcal{V} = 1, 2, \dots, \mathcal{Q})$. The AA ordered weighted geometric operator for $TpVFF$ numbers is defined in Definition 10, and Theorems 6–9 establish its mathematical properties.

Theorem 6. Let $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$, $\mathcal{V} = 1, 2, \dots, \mathcal{Q}$, be a collection of $TpVFF$ numbers. A mapping of $TpVFFAAOWG$ operator of dimension $\mathcal{Q}$ is a $TpVFFAAOWG : (T_p^*)^{\mathcal{Q}} \rightarrow (T_p^*)$ with a weight vector $\check{v}_{\mathcal{V}} = (\check{v}_1, \check{v}_2, \dots, \check{v}_{\mathcal{Q}})^T$ of $T_{p_{\mathcal{V}}}$ and $(\mathcal{V} = 1, 2, \dots, \mathcal{Q})$ with $\check{v}_{\mathcal{V}} \in [0, 1]$ and $\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}} = 1$. Then,

$$\begin{aligned} &= \left\langle \left(\mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}}(-\ln(a_{f_{\mathcal{V}\hat{a}}}))^Z\right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}}(-\ln(b_{f_{\mathcal{V}\hat{a}}}))^Z\right)^{\frac{1}{Z}}}, \right. \\ &\quad \left. \left(\mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}}(-\ln(c_{f_{\mathcal{V}\hat{a}}}))^Z\right)^{\frac{1}{Z}}}, \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}}(-\ln(d_{f_{\mathcal{V}\hat{a}}}))^Z\right)^{\frac{1}{Z}}}, \right. \right. \\ &\quad \left. \left(\sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}}(-\ln(1-w_{f_{\mathcal{V}\hat{b}}^3}))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}}(-\ln(1-x_{f_{\mathcal{V}\hat{b}}^3}))^Z\right)^{\frac{1}{Z}}}}, \right. \right. \\ &\quad \left. \left. \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}}(-\ln(1-y_{f_{\mathcal{V}\hat{b}}^3}))^Z\right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - \mathbf{e}^{-\left(\sum_{\mathcal{V}=1}^{\mathcal{Q}} \check{v}_{\mathcal{V}}(-\ln(1-z_{f_{\mathcal{V}\hat{b}}^3}))^Z\right)^{\frac{1}{Z}}}} \right) \right\rangle \end{aligned}$$

The $TpVFFAAOWG$ operator makes it simple to demonstrate the following properties, and because their proofs are similar to those of the preceding theorems, the proofs are omitted.

Theorem 7. Suppose that $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$ for all $\mathcal{V} = 1, 2, \dots, \mathcal{Q}$. Then, $TpVFFAAOWG(T_{p_1}, T_{p_2}, \dots, T_{p_Q}) = T_p$. (Idempotency)

Theorem 8. Let $T_{p_{\mathcal{V}}} = \langle (a_{\mathcal{V}\hat{a}}, b_{\mathcal{V}\hat{a}}, c_{\mathcal{V}\hat{a}}, d_{\mathcal{V}\hat{a}}), (w_{\mathcal{V}\hat{b}}, x_{\mathcal{V}\hat{b}}, y_{\mathcal{V}\hat{b}}, z_{\mathcal{V}\hat{b}}) \rangle$, $\mathcal{V} = 1, 2, \dots, \mathcal{Q}$, be a collection of $TpVFF$ numbers, and let $T_{p_{\mathcal{V}}}^- = \text{Min}\{T_{p_1}, T_{p_2}, \dots, T_{p_Q}\}$ and $T_{p_{\mathcal{V}}}^+ = \text{Max}\{T_{p_1}, T_{p_2}, \dots, T_{p_Q}\}$ denote their minimum and maximum values, respectively. Then, $T_{p_{\mathcal{V}}}^- \leq TpVFFAAOWG(T_{p_1}, T_{p_2}, \dots, T_{p_Q}) \leq T_{p_{\mathcal{V}}}^+$. (Boundedness)

Theorem 9. Let $T_{p\nu}$ and $\hat{T}_{p\nu}$, $\nu = 1, 2, \dots, Q$, be two collections of TpVFF numbers such that $T_{p\nu} \leq \hat{T}_{p\nu}$ for all ν . Then,

$$TpVFFAAOWG(T_{p1}, T_{p2}, \dots, T_{pQ}) \leq TpVFFAAOWG(\hat{T}_{p1}, \hat{T}_{p2}, \dots, \hat{T}_{pQ}).$$

(Monotonicity)

Definition 11. Let $T_{p\nu} = \langle (a_{\nu\hat{a}}, b_{\nu\hat{a}}, c_{\nu\hat{a}}, d_{\nu\hat{a}}), (w_{\nu\hat{b}}, x_{\nu\hat{b}}, y_{\nu\hat{b}}, z_{\nu\hat{b}}) \rangle$, $\nu = 1, 2, \dots, Q$, be a collection of TpVFF numbers. The $TpVFFAAHG$ operator is defined as the mapping $TpVFFAAHG : (Tp^*)^Q \rightarrow Tp^*$, with the weight vector $\check{v} = (\check{v}_1, \check{v}_2, \dots, \check{v}_Q)^T$, where $\check{v}_\nu \in [0, 1]$ and $\sum_{\nu=1}^Q \check{v}_\nu = 1$. It is defined as follows:

$$TpVFFAAHG(T_{p1}, T_{p2}, \dots, T_{pQ}) = \bigotimes_{\nu=1}^Q (\dot{T}_{pf(\nu)})^{\check{v}_\nu} = (\dot{T}_{pf(1)})^{\check{v}_1} \otimes (\dot{T}_{pf(2)})^{\check{v}_2} \otimes \dots \otimes (\dot{T}_{pf(Q)})^{\check{v}_Q}.$$

where $(\dot{T}_{pf(\nu)})^{\check{v}_\nu} = Q\check{v}_\nu(T_{pf(\nu)})^{\check{v}_\nu}$. The sequence $(\dot{T}_{pf(1)}^{\check{v}_1}, \dot{T}_{pf(2)}^{\check{v}_2}, \dots, \dot{T}_{pf(Q)}^{\check{v}_Q})$ is a permutation of $(T_{pf(1)}^{\check{v}_1}, T_{pf(2)}^{\check{v}_2}, \dots, T_{pf(Q)}^{\check{v}_Q})$ such that $\dot{T}_{pf(\nu-1)} \geq \dot{T}_{pf(\nu)}$, $\nu = 2, 3, \dots, Q$. The vector $\check{w} = (\check{w}_1, \check{w}_2, \dots, \check{w}_Q)^T$ is the weight vector, where $\check{w}_\nu \in [0, 1]$ and $\sum_{\nu=1}^Q \check{w}_\nu = 1$, and Q is the balancing coefficient.

The $\mathcal{AA}$ hybrid geometric operator for TpVFF numbers is defined in Definition 11, and Theorems 10–13 are used to examine the mathematical characteristics of $TpVFFAAHG$.

Theorem 10. Let $T_{p\nu} = \langle (a_{\nu\hat{a}}, b_{\nu\hat{a}}, c_{\nu\hat{a}}, d_{\nu\hat{a}}), (w_{\nu\hat{b}}, x_{\nu\hat{b}}, y_{\nu\hat{b}}, z_{\nu\hat{b}}) \rangle$, $\nu = 1, 2, \dots, Q$ be a collection of TpVFF numbers. A mapping of $TpVFFAAHG$ is a $TpVFFAAHGO$ of dimension $Q : (Tp^*)^Q \rightarrow (Tp^*)$ with a weight vector $\check{v}_\nu = (\check{v}_1, \check{v}_2, \dots, \check{v}_Q)^T$ of $T_{p\nu}$ and $(\nu = 1, 2, \dots, Q)$ with $\check{v}_\nu \in [0, 1]$ and $\sum_{\nu=1}^Q \check{v}_\nu = 1$. Then,

$$\begin{aligned} &= \left\langle \left(e^{-\left(\sum_{\nu=1}^Q \check{v}_\nu (-\ln(\dot{a}_{f\nu\hat{a}}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\sum_{\nu=1}^Q \check{v}_\nu (-\ln(\dot{b}_{f\nu\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \right. \\ &\quad \left. \left(e^{-\left(\sum_{\nu=1}^Q \check{v}_\nu (-\ln(\dot{c}_{f\nu\hat{a}}))^Z \right)^{\frac{1}{Z}}}, e^{-\left(\sum_{\nu=1}^Q \check{v}_\nu (-\ln(\dot{d}_{f\nu\hat{a}}))^Z \right)^{\frac{1}{Z}}}, \right. \right. \\ &\quad \left. \left(\sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^Q \check{v}_\nu (-\ln(1 - \dot{w}_{f\nu\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^Q \check{v}_\nu (-\ln(1 - \dot{x}_{f\nu\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \right. \right. \\ &\quad \left. \left. \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^Q \check{v}_\nu (-\ln(1 - \dot{y}_{f\nu\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}}, \sqrt[3]{1 - e^{-\left(\sum_{\nu=1}^Q \check{v}_\nu (-\ln(1 - \dot{z}_{f\nu\hat{b}}^3))^Z \right)^{\frac{1}{Z}}}} \right) \right\rangle \end{aligned}$$

The following properties of the $TpVFFAAHG$ operator can be established, and as their proofs are similar to those of previous theorems, they are omitted.

Theorem 11. Suppose that $T_{p\nu} = \langle (a_{\nu\hat{a}}, b_{\nu\hat{a}}, c_{\nu\hat{a}}, d_{\nu\hat{a}}), (w_{\nu\hat{b}}, x_{\nu\hat{b}}, y_{\nu\hat{b}}, z_{\nu\hat{b}}) \rangle$ for all $(\nu = 1, 2, \dots, Q)$. Then, $TpVFFAAHG(T_{p1}, T_{p2}, \dots, T_{pQ}) = T_p$. (Idempotency)

Theorem 12. Let $T_{p\nu} = \langle (a_{\nu\hat{a}}, b_{\nu\hat{a}}, c_{\nu\hat{a}}, d_{\nu\hat{a}}), (w_{\nu\hat{b}}, x_{\nu\hat{b}}, y_{\nu\hat{b}}, z_{\nu\hat{b}}) \rangle$, $(\nu = 1, 2, \dots, Q)$, be a collection of TpVFF numbers, and let $T_{p\nu}^- = \text{Min}\{T_{p1}, T_{p2}, \dots, T_{pQ}\}$ and $T_{p\nu}^+ = \text{Max}\{T_{p1}, T_{p2}, \dots, T_{pQ}\}$ denote their minimum and maximum values, respectively. Then, $T_{p\nu}^- \leq TpVFFAAHG(T_{p1}, T_{p2}, \dots, T_{pQ}) \leq T_{p\nu}^+$. (Boundedness)

Theorem 13. Let $T_{p\nu}$ and $\hat{T}_{p\nu}$, $\nu = 1, 2, \dots, Q$, be two collections of TpVFF numbers such that $T_{p\nu} \leq \hat{T}_{p\nu}$ for all ν . Then, $TpVFFAAHG(T_{p1}, T_{p2}, \dots, T_{pQ}) \leq TpVFFAAHG(\hat{T}_{p1}, \hat{T}_{p2}, \dots, \hat{T}_{pQ})$, where $T_{p\nu} = \langle (a_{\nu\hat{a}}, b_{\nu\hat{a}}, c_{\nu\hat{a}}, d_{\nu\hat{a}}), (w_{\nu\hat{b}}, x_{\nu\hat{b}}, y_{\nu\hat{b}}, z_{\nu\hat{b}}) \rangle$ and $\hat{T}_{p\nu} = \langle (\hat{a}_{\nu\hat{a}}, \hat{b}_{\nu\hat{a}}, \hat{c}_{\nu\hat{a}}, \hat{d}_{\nu\hat{a}}), (\hat{w}_{\nu\hat{b}}, \hat{x}_{\nu\hat{b}}, \hat{y}_{\nu\hat{b}}, \hat{z}_{\nu\hat{b}}) \rangle$. (Monotonicity)

5. Modified TOPSIS Method for Trapezoidal-Valued Fermatean Fuzzy Numbers

This section presents the updated TpVFF-TOPSIS approach for $\mathcal{MCGDM}$. The proposed technique incorporates the TpVFF number with the $\mathcal{AA}$ aggregation operation. The general procedure of the method is shown in Figure 1.

Let $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_i\}$ denote the set of alternatives and $\mathcal{Z} = \{\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_j\}$ the set of criteria. Let $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_p\}$ denote the set of decision makers and $d_K = (d_1, d_2, \dots, d_p)^K$ be weight of experts, then $d_p \geq 0$ and $(K = 1, 2, \dots, p)$ with $\sum_{K=1}^p d_K = 1$. Then, suppose that $\mathcal{M}^K = (\phi_{uv}^K)_{i \times j}$ is a TpVFF decision matrix, $\mathcal{E}_K$ ($K = 1, 2, \dots, p$) where $\phi_{uv}^K = \langle (a_{uv\mu}^K, b_{uv\mu}^K, c_{uv\mu}^K, d_{uv\mu}^K), (w_{uvv}^K, x_{uvv}^K, y_{uvv}^K, z_{uvv}^K) \rangle$ is a TpVFF number. Then, the criterion-weight vector is $\chi_d = (\chi_1, \chi_2, \dots, \chi_p)^d$ with $\chi_d \in [0, 1]$ and $\sum_{d=1}^p \chi_d = 1$.

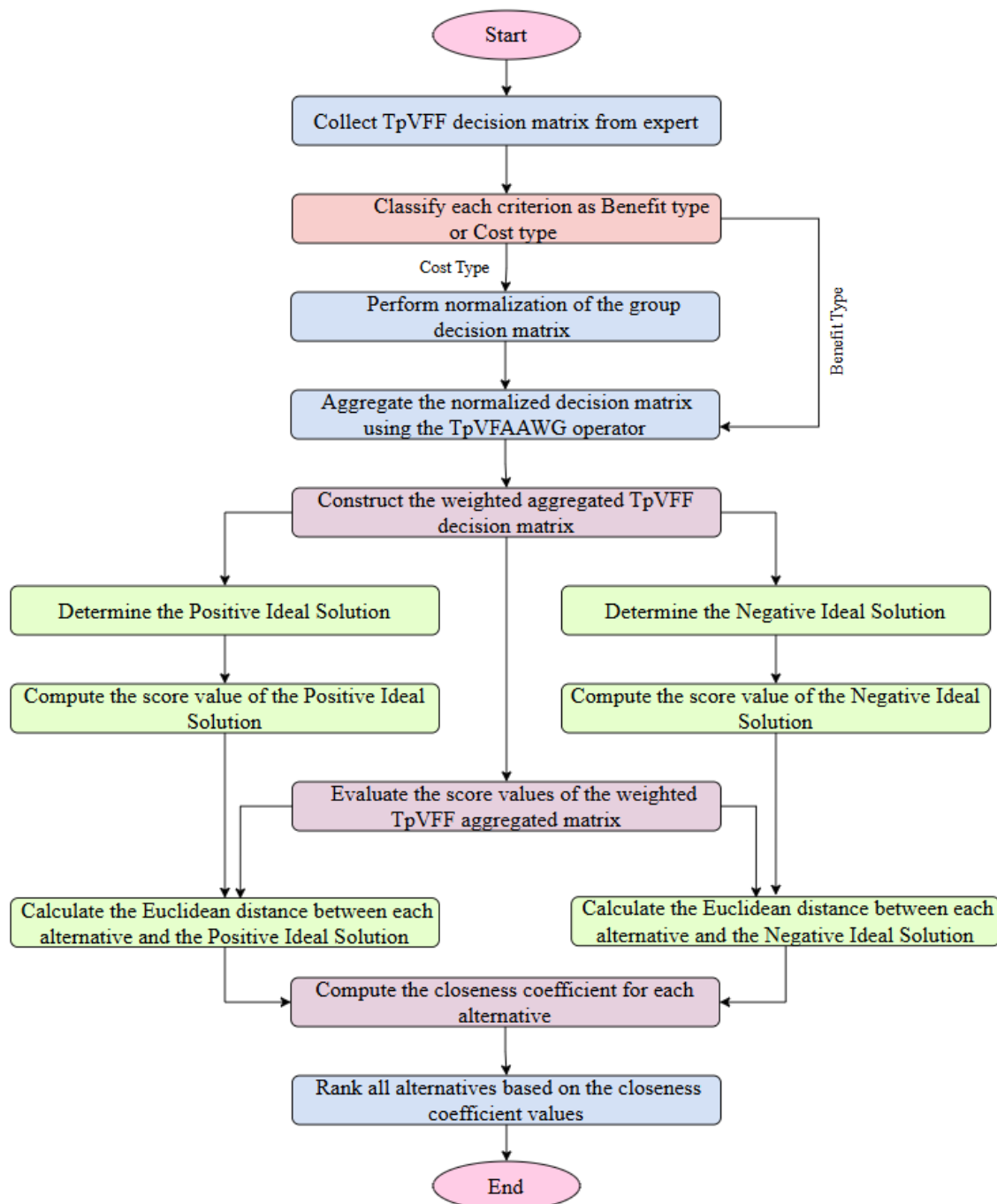


Fig. 1. TOPSIS Method

5.1 An Efficient Modified TOPSIS Using Trapezoidal-Valued Fermatean Fuzzy Numbers

The following steps describe the detailed procedure for using the modified TpVFF-TOPSIS method to solve the $\mathcal{MCGDM}$ problem. The corresponding flowchart is shown in Figure 1.

Step 1: Each decision expert $\mathfrak{P}_K$ provides a trapezoidal-valued Fermatean fuzzy evaluation matrix.

$$\mathfrak{M}^K = (\phi_{uv}^K)_{i \times j} = \left(\langle (a_{uv\mu}^K, b_{uv\mu}^K, c_{uv\mu}^K, d_{uv\mu}^K), (w_{uv\nu}^K, x_{uv\nu}^K, y_{uv\nu}^K, z_{uv\nu}^K) \rangle \right)_{i \times j}.$$

$0 \leq u \leq i, 0 \leq v \leq j$, and p denotes the number of experts.

Step 2: Each expert matrix is normalized. While the $\mathcal{M}$ and $\mathcal{NM}$ trapezoidal components are switched for cost type criteria, the normalized TpVFF number stays the same for benefit type criteria. The normalized group matrix is represented by

$$\mathfrak{M}' = (\phi_{uv}')_{i \times j} = \left(\langle (a_{uv\mu}, b_{uv\mu}, c_{uv\mu}, d_{uv\mu}), (w_{uv\nu}, x_{uv\nu}, y_{uv\nu}, z_{uv\nu}) \rangle \right)_{i \times j}.$$

$$(\phi_{uv}')_{i \times j} = \begin{cases} \langle (a_{uv\mu}, b_{uv\mu}, c_{uv\mu}, d_{uv\mu}), (w_{uv\nu}, x_{uv\nu}, y_{uv\nu}, z_{uv\nu}) \rangle & \text{benefit type} \\ \langle (w_{uv\nu}, x_{uv\nu}, y_{uv\nu}, z_{uv\nu}), (a_{uv\mu}, b_{uv\mu}, c_{uv\mu}, d_{uv\mu}) \rangle & \text{cost type} \end{cases}$$

Step 3: The $TpVFFAAWG$ operator combines all normalized expert matrices into a single collective decision matrix. That is, using equation 1.

Step 4: The combined decision matrix is weighted using the criterion weights χ_d . The weighting condition in Definition 8 (iii) is then applied to obtain the weighted TpVFF decision matrix.

Step 5: The positive ideal solution (PIS) and negative ideal solution (NIS) for the TpVFF decision matrix are defined as follows:

$$\mathbb{P}_{T_p}^+ = \{ \langle \mathfrak{J}_j, (max_{1 \leq u \leq i} a_{uv\mu}, max_{1 \leq u \leq i} b_{uv\mu}, max_{1 \leq u \leq i} c_{uv\mu}, max_{1 \leq u \leq i} d_{uv\mu}), (min_{1 \leq u \leq i} w_{uv\nu}, min_{1 \leq u \leq i} x_{uv\nu}, min_{1 \leq u \leq i} y_{uv\nu}, min_{1 \leq u \leq i} z_{uv\nu}) | \mathfrak{J}_j \in \mathfrak{J} \rangle \} \quad (2)$$

$$\mathbb{N}_{T_p}^- = \{ \langle \mathfrak{J}_j, (min_{1 \leq u \leq i} a_{uv\mu}, min_{1 \leq u \leq i} b_{uv\mu}, min_{1 \leq u \leq i} c_{uv\mu}, min_{1 \leq u \leq i} d_{uv\mu}), (max_{1 \leq u \leq i} w_{uv\nu}, max_{1 \leq u \leq i} x_{uv\nu}, max_{1 \leq u \leq i} y_{uv\nu}, max_{1 \leq u \leq i} z_{uv\nu}) | \mathfrak{J}_j \in \mathfrak{J} \rangle \} \quad (3)$$

Step 6: Let $S(\phi'_{uv})$ denote the scalar score value of the weighted $TpVFFN$ ϕ'_{uv} , and let $S(\mathbb{P}_{T_p}^+)$ and $S(\mathbb{N}_{T_p}^-)$ denote the scalar score values of the PIS and NIS, respectively. The score values of the weighted decision matrix and the corresponding ideal solutions are calculated before the Euclidean distances are determined.

Step 7: The Euclidean distances between each alternative and the PIS, and between each alternative and the NIS, are calculated using the scalar score values obtained in Step 6. Similarly, the distance of the u th alternative $\mathfrak{B}_u$ from the positive ideal solution is defined as

$$\mathbb{D}(\mathbb{P}_{T_p}^+, \mathfrak{B}_u) = \sqrt{\sum_{v=1}^j [S(\phi'_{uv}) - S(\mathbb{P}_{T_p}^+, v)]^2}. \quad (4)$$

Similarly, the distance of the u th alternative $\mathfrak{B}_u$ from the negative ideal solution is defined as

$$\mathbb{D}(\mathbb{N}_{T_p}^-, \mathfrak{B}_u) = \sqrt{\sum_{v=1}^j [S(\phi'_{uv}) - S(\mathbb{N}_{T_p}^-, v)]^2}. \quad (5)$$

Here, $S(\phi'_{uv})$ denotes the scalar score value of the weighted $TpVFFN$ corresponding to the u th alternative and the v th criterion, while $S(\mathbb{P}_{T_p}^+, v)$ and $S(\mathbb{N}_{T_p}^-, v)$ denote the scalar score values of the positive and negative ideal solutions, respectively, with respect to the v th criterion.

Step 8: The relative closeness coefficient ($\mathbb{CC}_{T_{p_i}}(\mathfrak{B}_u)$) for each alternative is calculated using Equation 6

$$\mathbb{CC}_{T_{p_i}}(\mathfrak{B}_u) = \frac{\mathbb{D}(\mathbb{N}_{T_p}^-, \mathfrak{B}_u)}{\mathbb{D}(\mathbb{P}_{T_p}^+, \mathfrak{B}_u) + \mathbb{D}(\mathbb{N}_{T_p}^-, \mathfrak{B}_u)}. \quad (6)$$

Step 9: A larger value of $\mathbb{CC}_{T_{p_i}}(\mathfrak{B}_u)$ indicates that the corresponding alternative is closer to the

positive ideal solution and farther from the negative ideal solution. The alternatives are ranked in descending order of their closeness coefficients.

5.2 MCGDM Application for Disaster Relief Allocation Problem

Let $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_i\}$ denote the set of alternatives, $\mathcal{Z} = \{\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_j\}$ denote the set of criteria, and $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_p\}$ denote the set of decision makers, where i, j , and p represent the numbers of alternatives, criteria, and decision makers, respectively. To illustrate the application of the proposed TpVFF-TOPSIS framework, a hypothetical scenario involving five criteria is considered. These criteria include damage severity, local resource availability, population affected, urgency of medical needs, and region accessibility. The illustrative scenario is hypothetical and is not based on any particular real-world disaster.

Disaster assessment involves subjectivity, ambiguity, and uncertainty because of incomplete information and rapidly changing circumstances. The TpVFF framework provides a representation of uncertainty based on the Fermatean condition, in which the sum of the cubes of the $\mathcal{M}$ and $\mathcal{NM}$ degrees is bounded. This formulation allows uncertain assessments to be represented within the proposed framework.

To rank the impacted areas and establish the priority order for the effective and fair allocation of scarce relief resources, the suggested TpVFF-TOPSIS technique is applied. In the $\mathcal{DM}$ problem, five disaster-affected locations represented by $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$, and $\mathcal{B}_5$ are taken into consideration as alternatives. The five criteria considered in this hypothetical numerical illustration are selected based on their relevance to disaster relief allocation. They are Damage severity ($\mathcal{Z}_1$), local resource availability ($\mathcal{Z}_2$), population affected ($\mathcal{Z}_3$), urgency of medical needs ($\mathcal{Z}_4$), and region accessibility ($\mathcal{Z}_5$). The purpose of the present study is to demonstrate the proposed method using an illustrative decision-making matrix. Thus, the expert evaluations and their respective weights are assumed only for illustrative purposes and are not derived from actual expert elicitation or survey data. Accordingly, let the expert-weight vector be $(0.13, 0.17, 0.18, 0.16, 0.19, 0.17)$ and the criterion weight vector be $(0.17, 0.23, 0.18, 0.25, 0.17)$

Damage severity $\mathcal{Z}_1$: This criterion assesses the degree of devastation brought about by the disaster, taking into account harm to residential areas, infrastructure, and vital public services. A higher severity level indicates a greater need for relief intervention.

Local resource availability $\mathcal{Z}_2$: This criterion evaluates the availability of essential local resources, such as food, water, and shelter, from the perspective of external relief requirements. A higher linguistic assessment indicates a greater need for external relief assistance because of limited local resources.

Population affected $\mathcal{Z}_3$: The number of persons affected by the disaster is reflected in this criterion. Areas with larger affected populations are assigned a higher priority for relief resources.

Urgency of medical needs $\mathcal{Z}_4$: This criterion assesses the urgent need for medical treatment, taking into account illness outbreaks, injuries, and the accessibility of medical facilities. Higher urgency corresponds to situations requiring rapid medical intervention.

Region accessibility $\mathcal{Z}_5$: This criterion evaluates the accessibility of the affected area from the perspective of relief delivery requirements. A higher linguistic assessment indicates a greater need for relief intervention because of accessibility constraints and associated logistical difficulties.

In the numerical example, $m = 5$ alternatives, $n = 5$ criteria, and $p = 6$ decision makers are considered. Thus, the proposed TpVFF-TOPSIS framework is applied to determine the priority ranking of the five hypothetical disaster-affected regions under uncertain conditions using the assumed expert evaluations expressed in TpVFF number. The example serves as a numerical illustration of the proposed decision-making framework.

The relationship between qualitative evaluations and the corresponding TpVFF numbers is established using the linguistic scale shown in Table 1. Transforming linguistic concepts into a mathematically tractable form makes it possible to handle the ambiguity and vagueness inherent in qualitative assessments. The established scale is used for the subsequent aggregation and decision-making pro-

cedures.

Table 1
Linguistic Scale

Linguistic Variables	$TpVFFNs$
Emergency Priority ($\mathcal{EP}$)	$\langle (0.67, 0.72, 0.77, 0.82), (0.52, 0.57, 0.62, 0.67) \rangle$
Critical Priority ($\mathcal{CP}$)	$\langle (0.62, 0.67, 0.72, 0.77), (0.57, 0.62, 0.67, 0.72) \rangle$
High Priority ($\mathcal{HP}$)	$\langle (0.57, 0.62, 0.67, 0.72), (0.62, 0.67, 0.72, 0.77) \rangle$
Moderate Priority ($\mathcal{MP}$)	$\langle (0.52, 0.57, 0.62, 0.67), (0.67, 0.72, 0.77, 0.82) \rangle$
Low Priority ($\mathcal{LP}$)	$\langle (0.47, 0.52, 0.57, 0.62), (0.72, 0.77, 0.82, 0.87) \rangle$
Negligible Priority ($\mathcal{NP}$)	$\langle (0.42, 0.47, 0.52, 0.57), (0.77, 0.82, 0.87, 0.92) \rangle$

These evaluations represent assumed qualitative assessments used for numerical illustration, which are then converted using the predetermined language scale into TpVFF numbers in Table 2. The resulting assessment data serve as the input for the aggregation process.

Table 2
Assumed linguistic evaluations for the numerical illustration

$\mathcal{E}_1$						$\mathcal{E}_2$					
	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$		$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$
$\mathcal{B}_1$	$\mathcal{EP}$	$\mathcal{CP}$	$\mathcal{MP}$	$\mathcal{HP}$	$\mathcal{EP}$	$\mathcal{B}_1$	$\mathcal{HP}$	$\mathcal{CP}$	$\mathcal{MP}$	$\mathcal{NP}$	$\mathcal{LP}$
$\mathcal{B}_2$	$\mathcal{CP}$	$\mathcal{NP}$	$\mathcal{CP}$	$\mathcal{LP}$	$\mathcal{EP}$	$\mathcal{B}_2$	$\mathcal{EP}$	$\mathcal{HP}$	$\mathcal{HP}$	$\mathcal{MP}$	$\mathcal{LP}$
$\mathcal{B}_3$	$\mathcal{MP}$	$\mathcal{MP}$	$\mathcal{LP}$	$\mathcal{EP}$	$\mathcal{HP}$	$\mathcal{B}_3$	$\mathcal{MP}$	$\mathcal{MP}$	$\mathcal{CP}$	$\mathcal{NP}$	$\mathcal{EP}$
$\mathcal{B}_4$	$\mathcal{HP}$	$\mathcal{MP}$	$\mathcal{LP}$	$\mathcal{EP}$	$\mathcal{HP}$	$\mathcal{B}_4$	$\mathcal{LP}$	$\mathcal{NP}$	$\mathcal{HP}$	$\mathcal{MP}$	$\mathcal{CP}$
$\mathcal{B}_5$	$\mathcal{CP}$	$\mathcal{EP}$	$\mathcal{NP}$	$\mathcal{HP}$	$\mathcal{NP}$	$\mathcal{B}_5$	$\mathcal{NP}$	$\mathcal{MP}$	$\mathcal{LP}$	$\mathcal{NP}$	$\mathcal{MP}$
$\mathcal{E}_3$						$\mathcal{E}_4$					
	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$		$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$
$\mathcal{B}_1$	$\mathcal{MP}$	$\mathcal{NP}$	$\mathcal{HP}$	$\mathcal{MP}$	$\mathcal{CP}$	$\mathcal{B}_1$	$\mathcal{EP}$	$\mathcal{CP}$	$\mathcal{HP}$	$\mathcal{LP}$	$\mathcal{MP}$
$\mathcal{B}_2$	$\mathcal{NP}$	$\mathcal{MP}$	$\mathcal{EP}$	$\mathcal{NP}$	$\mathcal{MP}$	$\mathcal{B}_2$	$\mathcal{CP}$	$\mathcal{HP}$	$\mathcal{MP}$	$\mathcal{NP}$	$\mathcal{NP}$
$\mathcal{B}_3$	$\mathcal{HP}$	$\mathcal{LP}$	$\mathcal{LP}$	$\mathcal{LP}$	$\mathcal{NP}$	$\mathcal{B}_3$	$\mathcal{MP}$	$\mathcal{HP}$	$\mathcal{MP}$	$\mathcal{NP}$	$\mathcal{MP}$
$\mathcal{B}_4$	$\mathcal{MP}$	$\mathcal{MP}$	$\mathcal{MP}$	$\mathcal{EP}$	$\mathcal{LP}$	$\mathcal{B}_4$	$\mathcal{NP}$	$\mathcal{NP}$	$\mathcal{NP}$	$\mathcal{LP}$	$\mathcal{LP}$
$\mathcal{B}_5$	$\mathcal{LP}$	$\mathcal{EP}$	$\mathcal{CP}$	$\mathcal{EP}$	$\mathcal{EP}$	$\mathcal{B}_5$	$\mathcal{NP}$	$\mathcal{LP}$	$\mathcal{LP}$	$\mathcal{HP}$	$\mathcal{CP}$
$\mathcal{E}_5$						$\mathcal{E}_6$					
	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$		$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$
$\mathcal{B}_1$	$\mathcal{EP}$	$\mathcal{LP}$	$\mathcal{CP}$	$\mathcal{HP}$	$\mathcal{MP}$	$\mathcal{B}_1$	$\mathcal{EP}$	$\mathcal{LP}$	$\mathcal{NP}$	$\mathcal{MP}$	$\mathcal{NP}$
$\mathcal{B}_2$	$\mathcal{CP}$	$\mathcal{EP}$	$\mathcal{HP}$	$\mathcal{NP}$	$\mathcal{MP}$	$\mathcal{B}_2$	$\mathcal{CP}$	$\mathcal{CP}$	$\mathcal{CP}$	$\mathcal{MP}$	$\mathcal{NP}$
$\mathcal{B}_3$	$\mathcal{HP}$	$\mathcal{HP}$	$\mathcal{EP}$	$\mathcal{LP}$	$\mathcal{LP}$	$\mathcal{B}_3$	$\mathcal{HP}$	$\mathcal{CP}$	$\mathcal{LP}$	$\mathcal{CP}$	$\mathcal{EP}$
$\mathcal{B}_4$	$\mathcal{LP}$	$\mathcal{MP}$	$\mathcal{CP}$	$\mathcal{MP}$	$\mathcal{EP}$	$\mathcal{B}_4$	$\mathcal{EP}$	$\mathcal{EP}$	$\mathcal{HP}$	$\mathcal{EP}$	$\mathcal{MP}$
$\mathcal{B}_5$	$\mathcal{NP}$	$\mathcal{LP}$	$\mathcal{NP}$	$\mathcal{CP}$	$\mathcal{NP}$	$\mathcal{B}_5$	$\mathcal{NP}$	$\mathcal{HP}$	$\mathcal{LP}$	$\mathcal{HP}$	$\mathcal{EP}$

Here, the linguistic assessments do not represent the desirability of the raw criterion values; instead, they represent the priority or need for disaster relief associated with each criterion. For the disaster relief allocation problem, all five criteria are assumed to be benefit criteria. The linguistic assessments are assigned from the perspective of disaster relief allocation priorities, so that a higher level of assessment means a higher priority for allocation of the disaster relief. Consequently, the benefit type normalization method is applied to all five criteria. The aggregated decision matrix obtained by combining the assumed linguistic assessments using the proposed aggregation operator is

presented in Table 3. The resulting TpVFF numbers are used in the subsequent numerical analysis and represent the aggregated evaluations of the alternatives under each criterion. The weighted decision

Table 3
Aggregated Matrix Using the TpVFFAAWG Operator for $\mathbb{Z} = 2$

$\mathcal{Z}_1$	
$\mathcal{B}_1$	$\langle (0.615881, 0.665465, 0.714724, 0.763502), (0.586140, 0.635858, 0.686150, 0.737207) \rangle$
$\mathcal{B}_2$	$\langle (0.572405, 0.622160, 0.671388, 0.719936), (0.645453, 0.697024, 0.751031, 0.809743) \rangle$
$\mathcal{B}_3$	$\langle (0.545485, 0.595466, 0.645414, 0.695317), (0.646198, 0.696239, 0.746382, 0.796679) \rangle$
$\mathcal{B}_4$	$\langle (0.503459, 0.553399, 0.603063, 0.652397), (0.699034, 0.749829, 0.801611, 0.855269) \rangle$
$\mathcal{B}_5$	$\langle (0.446102, 0.496174, 0.546103, 0.595875), (0.750327, 0.800909, 0.852031, 0.904151) \rangle$
$\mathcal{Z}_2$	
$\mathcal{B}_1$	$\langle (0.512896, 0.562849, 0.612461, 0.661665), (0.693592, 0.744637, 0.796978, 0.851722) \rangle$
$\mathcal{B}_2$	$\langle (0.554188, 0.604021, 0.653501, 0.702533), (0.653712, 0.704754, 0.757385, 0.813198) \rangle$
$\mathcal{B}_3$	$\langle (0.539715, 0.589656, 0.639471, 0.689125), (0.656612, 0.706853, 0.757525, 0.808906) \rangle$
$\mathcal{B}_4$	$\langle (0.496792, 0.546785, 0.596481, 0.645831), (0.707246, 0.758366, 0.810743, 0.865607) \rangle$
$\mathcal{B}_5$	$\langle (0.541776, 0.591522, 0.640927, 0.689896), (0.662363, 0.712711, 0.763863, 0.816259) \rangle$
$\mathcal{Z}_3$	
$\mathcal{B}_1$	$\langle (0.528337, 0.578333, 0.628095, 0.677577), (0.674802, 0.725864, 0.778245, 0.833346) \rangle$
$\mathcal{B}_2$	$\langle (0.589331, 0.639148, 0.688811, 0.738253), (0.606711, 0.656635, 0.706853, 0.757475) \rangle$
$\mathcal{B}_3$	$\langle (0.526565, 0.576383, 0.625897, 0.675032), (0.676151, 0.726538, 0.777649, 0.829877) \rangle$
$\mathcal{B}_4$	$\langle (0.522110, 0.572106, 0.621860, 0.671326), (0.681020, 0.732029, 0.784276, 0.839004) \rangle$
$\mathcal{B}_5$	$\langle (0.470960, 0.521005, 0.570877, 0.620552), (0.727221, 0.777908, 0.829307, 0.882093) \rangle$
$\mathcal{Z}_4$	
$\mathcal{B}_1$	$\langle (0.503833, 0.553883, 0.603786, 0.653523), (0.694636, 0.745453, 0.797175, 0.850788) \rangle$
$\mathcal{B}_2$	$\langle (0.455502, 0.505594, 0.555594, 0.605499), (0.739657, 0.790317, 0.841587, 0.894124) \rangle$
$\mathcal{B}_3$	$\langle (0.486562, 0.536547, 0.586202, 0.635474), (0.717954, 0.768945, 0.821043, 0.875186) \rangle$
$\mathcal{B}_4$	$\langle (0.566901, 0.616495, 0.665674, 0.714296), (0.639609, 0.689765, 0.740791, 0.793122) \rangle$
$\mathcal{B}_5$	$\langle (0.555518, 0.605366, 0.654814, 0.703759), (0.655479, 0.706813, 0.760066, 0.817141) \rangle$
$\mathcal{Z}_5$	
$\mathcal{B}_1$	$\langle (0.516501, 0.566423, 0.616032, 0.665264), (0.688241, 0.739186, 0.791346, 0.845861) \rangle$
$\mathcal{B}_2$	$\langle (0.484760, 0.534778, 0.584557, 0.634060), (0.716431, 0.767358, 0.819288, 0.873200) \rangle$
$\mathcal{B}_3$	$\langle (0.528036, 0.577836, 0.627178, 0.675954), (0.682464, 0.733570, 0.786222, 0.841760) \rangle$
$\mathcal{B}_4$	$\langle (0.541218, 0.591016, 0.640515, 0.689631), (0.661755, 0.712117, 0.763246, 0.815578) \rangle$
$\mathcal{B}_5$	$\langle (0.528694, 0.578494, 0.627705, 0.676198), (0.688146, 0.739752, 0.793429, 0.850934) \rangle$

matrix, which adjusts the aggregated judgments according to the relative importance of the criteria, is shown in Table 4. This transformation guarantees that the total assessment of alternatives adequately reflects the contribution of each criterion.

The score function is used solely to rank the aggregated TpVFF numbers. The score function preserves the ordering of the TpVFF numbers and facilitates the ranking procedure in the TOPSIS method. The calculated score values of the options are shown in Table 5. These score values facilitate the comparison and ranking of the alternatives within the proposed decision-making framework.

The results demonstrate the applicability of the proposed framework for ranking alternatives in a TpVFF-based $\mathcal{MCGDM}$ environment. However, the sensitivity analysis indicates that the obtained ranking depends on the selected value of the $\mathcal{AA}$ parameter. In particular, the highest-ranked alternative changes across the tested values of $\mathbb{Z}$, indicating that the ranking should not be regarded as invariant with respect to the parameter choice. The Monte Carlo analysis further shows that variations in the criterion weights can also alter the relative positions of the alternatives. Therefore, the

Table 4
Weighted Aggregated Matrix

	$\mathcal{Z}_1$
$\mathcal{B}_1$	$\langle(0.470113, 0.511810, 0.554847, 0.599615), (0.802314, 0.829704, 0.856158, 0.881872)\rangle$
$\mathcal{B}_2$	$\langle(0.434574, 0.475318, 0.516890, 0.559517), (0.834844, 0.861726, 0.888653, 0.916665)\rangle$
$\mathcal{B}_3$	$\langle(0.412954, 0.453321, 0.494777, 0.537668), (0.835241, 0.861326, 0.886381, 0.910538)\rangle$
$\mathcal{B}_4$	$\langle(0.379682, 0.419282, 0.459547, 0.500681), (0.862750, 0.888067, 0.912858, 0.937573)\rangle$
$\mathcal{B}_5$	$\langle(0.335033, 0.373966, 0.413447, 0.453655), (0.888310, 0.912528, 0.936108, 0.959307)\rangle$
	$\mathcal{Z}_2$
$\mathcal{B}_1$	$\langle(0.406455, 0.447953, 0.490023, 0.532845), (0.839066, 0.868134, 0.896882, 0.925917)\rangle$
$\mathcal{B}_2$	$\langle(0.440702, 0.482795, 0.525651, 0.569490), (0.815572, 0.845515, 0.875230, 0.905590)\rangle$
$\mathcal{B}_3$	$\langle(0.428640, 0.470564, 0.513374, 0.557343), (0.817305, 0.846722, 0.875308, 0.903295)\rangle$
$\mathcal{B}_4$	$\langle(0.393227, 0.434524, 0.476364, 0.518926), (0.846948, 0.875774, 0.904278, 0.933126)\rangle$
$\mathcal{B}_5$	$\langle(0.430354, 0.472148, 0.514644, 0.558038), (0.820731, 0.850080, 0.878812, 0.907223)\rangle$
	$\mathcal{Z}_3$
$\mathcal{B}_1$	$\langle(0.403008, 0.443391, 0.484584, 0.526842), (0.846303, 0.872903, 0.899093, 0.925570)\rangle$
$\mathcal{B}_2$	$\langle(0.452400, 0.493898, 0.536657, 0.581070), (0.808960, 0.836560, 0.863129, 0.888833)\rangle$
$\mathcal{B}_3$	$\langle(0.401592, 0.441799, 0.482740, 0.524631), (0.847020, 0.873247, 0.898801, 0.923933)\rangle$
$\mathcal{B}_4$	$\langle(0.398036, 0.438312, 0.479358, 0.521419), (0.849602, 0.876041, 0.902042, 0.928230)\rangle$
$\mathcal{B}_5$	$\langle(0.357602, 0.397155, 0.437311, 0.478265), (0.873595, 0.898928, 0.923664, 0.948165)\rangle$
	$\mathcal{Z}_4$
$\mathcal{B}_1$	$\langle(0.404400, 0.446332, 0.488942, 0.532449), (0.833448, 0.863396, 0.892847, 0.922382)\rangle$
$\mathcal{B}_2$	$\langle(0.364498, 0.405864, 0.447779, 0.490422), (0.860033, 0.888998, 0.917381, 0.945581)\rangle$
$\mathcal{B}_3$	$\langle(0.390082, 0.431727, 0.473823, 0.516523), (0.847322, 0.876895, 0.906114, 0.935514)\rangle$
$\mathcal{B}_4$	$\langle(0.457362, 0.499950, 0.543270, 0.587505), (0.799756, 0.830521, 0.860692, 0.890574)\rangle$
$\mathcal{B}_5$	$\langle(0.447714, 0.490307, 0.533595, 0.577779), (0.809617, 0.840722, 0.871818, 0.903959)\rangle$
	$\mathcal{Z}_5$
$\mathcal{B}_1$	$\langle(0.389950, 0.429747, 0.470238, 0.511638), (0.857233, 0.882848, 0.908020, 0.933307)\rangle$
$\mathcal{B}_2$	$\langle(0.365039, 0.404425, 0.444425, 0.485239), (0.871539, 0.896568, 0.921105, 0.945629)\rangle$
$\mathcal{B}_3$	$\langle(0.399073, 0.438970, 0.479492, 0.520823), (0.854259, 0.880076, 0.905591, 0.931439)\rangle$
$\mathcal{B}_4$	$\langle(0.409550, 0.449686, 0.490653, 0.532692), (0.843474, 0.869371, 0.894584, 0.919383)\rangle$
$\mathcal{B}_5$	$\langle(0.399595, 0.439503, 0.479932, 0.521034), (0.857184, 0.883126, 0.909005, 0.935611)\rangle$

Table 5
Score Value and Ideal Solutions

	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$
$\mathcal{B}_1$	-0.222083	-0.291784	-0.297292	-0.287023	-0.312888
$\mathcal{B}_2$	-0.273882	-0.253999	-0.237031	-0.329153	-0.336621
$\mathcal{B}_3$	-0.279108	-0.258660	-0.297563	-0.307665	-0.307020
$\mathcal{B}_4$	-0.322053	-0.305062	-0.302518	-0.230570	-0.290162
$\mathcal{B}_5$	-0.363752	-0.262189	-0.341518	-0.246725	-0.311000
$S(\mathcal{V}_{\mathcal{T}_p}^+)$	-0.222083	-0.253999	-0.237031	-0.230570	-0.290162
$S(\mathcal{N}_{\mathcal{T}_p}^-)$	-0.363752	-0.305062	-0.341518	-0.329153	-0.336621

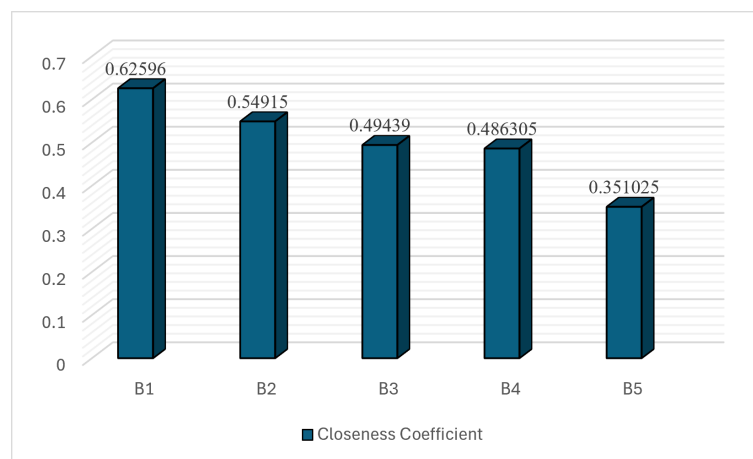
proposed framework provides a systematic decision-making procedure. The resulting rankings should be interpreted in relation to the selected parameter and criterion-weight configuration.

Table 6

Distances of alternatives from PIS and NIS, closeness coefficients, and final ranking results

	$\mathcal{D}(\mathcal{V}_{\mathcal{T}_p}^+, \mathfrak{B}_j)$	$\mathcal{D}(\mathcal{N}_{\mathcal{T}_p}^-, \mathfrak{B}_j)$	$\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)$	Final Ranking
$\mathcal{B}_1$	0.093608	0.156654	0.625963	1
$\mathcal{B}_2$	0.120666	0.146975	0.549150	2
$\mathcal{B}_3$	0.114741	0.112195	0.494390	3
$\mathcal{B}_4$	0.129961	0.123032	0.486305	4
$\mathcal{B}_5$	0.178185	0.096379	0.351025	5

The final ranking obtained from the proposed TpVFF-TOPSIS method is shown in Figure 2.

**Fig. 2.** Final ranking from TpVFF-TOPSIS method

For $\mathbb{Z} = 2$ and the specified criterion weights, alternative $\mathcal{B}_1$ has the highest closeness coefficient and is therefore ranked first in the considered numerical illustration. This result is obtained from the calculated closeness coefficient under the specified parameter and criterion-weight configuration. The sensitivity analysis shows that the resulting ranking varies with the value of the $\mathcal{AA}$ parameter $\mathbb{Z}$. Specifically, $\mathcal{B}_4$ is ranked first for $\mathbb{Z} = 0.5$, whereas $\mathcal{B}_1$ is ranked first for $\mathbb{Z} = 1$, $\mathbb{Z} = 2$, and $\mathbb{Z} = 5$. For $\mathbb{Z} = 10$, $\mathcal{B}_3$ becomes the highest-ranked alternative. Thus, the ranking is sensitive to the choice of $\mathbb{Z}$, and $\mathcal{B}_1$ does not maintain the leading position across all tested parameter values. The obtained results illustrate the applicability of the proposed approach for identifying and ranking alternatives within a TpVFF-based $\mathcal{MCGDM}$ environment. The calculation confirms the consistency between the scalar-weighting formula and the numerical procedure. In particular, the scalar-weighted $TpVFFN$ obtained from Definition 8(iii) produces the score $S = -0.222083$, which agrees with the corresponding entry in Table 5 and subsequently leads to $\mathbb{D}(\mathcal{P}_{\mathcal{T}_p}^+, \mathcal{B}_1) = 0.093608$, $\mathbb{D}(\mathcal{N}_{\mathcal{T}_p}^-, \mathcal{B}_1) = 0.156654$, and $\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathcal{B}_1) = 0.625963$ in Table 6. To illustrate the reproducibility of the numerical implementation, Table 7 presents the calculation for alternative $\mathcal{B}_1$ under criterion $\mathbb{Z}_1$, with the $\mathcal{AA}$ parameter set to $\mathbb{Z} = 2$. The scalar weighting in Table 8 is calculated using Definition 8(iii), with the exponent $\zeta^{1/\mathbb{Z}}$, to maintain consistency between the theoretical scalar operation and its numerical implementation.

6. Sensitivity Analysis

The sensitivity analysis examines the impact of the $\mathcal{AA}$ parameter $\mathbb{Z}$ on the decision results. To examine the effect of different values of $\mathbb{Z}$, we consider $\mathbb{Z} = 0.5, 1, 2, 5$, and 10 while keeping the

Table 7
Reproducibility check for $\mathcal{B}_1$ under $\mathcal{Z}_1$ with $\mathbb{Z} = 2$

Step	Calculation	Result
Original assessments	$\mathcal{EP}, \mathcal{HP}, \mathcal{MP}, \mathcal{EP}, \mathcal{EP}, \mathcal{EP}$	$\mathcal{EP}, \mathcal{HP}, \mathcal{MP}, \mathcal{EP}, \mathcal{EP}, \mathcal{EP}$
Expert aggregation	$TpVFFAAWG, \mathbb{Z} = 2$	$\langle (0.615881, 0.665465, 0.714724, 0.763502), (0.586140, 0.635858, 0.686150, 0.737207) \rangle$
Scalar weighting	$\chi_1 = 0.17, \chi_1^{1/\mathbb{Z}} = 0.41231056$	$\langle (0.470113, 0.511810, 0.554847, 0.599615), (0.802314, 0.829704, 0.856158, 0.881872) \rangle$
Score S	Using Definition 7	-0.222083
PIS	$S(\mathbb{P}_{T_p}^+)$	$(-0.222083, -0.253999, -0.237031, -0.230570, -0.290162)$
NIS	$S(\mathbb{N}_{T_p}^-)$	$(-0.363752, -0.305062, -0.341518, -0.329153, -0.336621)$
Distance to PIS	$\mathbb{D}(\mathbb{P}_{T_p}^+, \mathcal{B}_1)$	0.093608
Distance to NIS	$\mathbb{D}(\mathbb{N}_{T_p}^-, \mathcal{B}_1)$	0.156654
Closeness coefficient	$\frac{\mathbb{D}(\mathbb{N}_{T_p}^-, \mathcal{B}_u)}{\mathbb{D}(\mathbb{P}_{T_p}^+, \mathcal{B}_u) + \mathbb{D}(\mathbb{N}_{T_p}^-, \mathcal{B}_u)}$	$\frac{0.093608 + 0.156654}{0.625963} =$
Final rank	Descending order of $\mathbb{CC}_{T_{p_i}}(\mathcal{B}_u) =$	$\mathcal{B}_1$: Rank 1

criterion weights fixed. The complete $\mathcal{MCGDM}$ procedure is repeated for each value of $\mathbb{Z}$. The results obtained are given in Table 8.

Table 8
Sensitivity analysis with different values of the Aczel–Alsina parameter $\mathbb{Z}$

$\mathbb{Z}$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$	Ranking
0.5	0.535138	0.453283	0.423521	0.598455	0.516643	$\mathcal{B}_4 > \mathcal{B}_1 > \mathcal{B}_5 > \mathcal{B}_2 > \mathcal{B}_3$
1	0.602856	0.527084	0.464317	0.529109	0.416984	$\mathcal{B}_1 > \mathcal{B}_4 > \mathcal{B}_2 > \mathcal{B}_3 > \mathcal{B}_5$
2	0.625963	0.549150	0.494390	0.486305	0.351025	$\mathcal{B}_1 > \mathcal{B}_2 > \mathcal{B}_3 > \mathcal{B}_4 > \mathcal{B}_5$
5	0.574016	0.505142	0.531559	0.424768	0.266417	$\mathcal{B}_1 > \mathcal{B}_3 > \mathcal{B}_2 > \mathcal{B}_4 > \mathcal{B}_5$
10	0.508589	0.462820	0.555100	0.381975	0.232399	$\mathcal{B}_3 > \mathcal{B}_1 > \mathcal{B}_2 > \mathcal{B}_4 > \mathcal{B}_5$

The results indicate that the final ranking depends on the selected $\mathcal{AA}$ parameter. When $\mathbb{Z} = 0.5$, the ranking is $\mathcal{B}_4 > \mathcal{B}_1 > \mathcal{B}_5 > \mathcal{B}_2 > \mathcal{B}_3$, with $\mathcal{B}_4$ obtaining the highest closeness coefficient of 0.598455. For $\mathbb{Z} = 1$, the ranking changes to $\mathcal{B}_1 > \mathcal{B}_4 > \mathcal{B}_2 > \mathcal{B}_3 > \mathcal{B}_5$. At the baseline value $\mathbb{Z} = 2$, the ranking is $\mathcal{B}_1 > \mathcal{B}_2 > \mathcal{B}_3 > \mathcal{B}_4 > \mathcal{B}_5$, with $\mathcal{B}_1$ obtaining the highest closeness coefficient of 0.625963. When $\mathbb{Z} = 5$, the ranking becomes $\mathcal{B}_1 > \mathcal{B}_3 > \mathcal{B}_2 > \mathcal{B}_4 > \mathcal{B}_5$, whereas for $\mathbb{Z} = 10$, the ranking changes to $\mathcal{B}_3 > \mathcal{B}_1 > \mathcal{B}_2 > \mathcal{B}_4 > \mathcal{B}_5$. These results indicate that changes in the $\mathcal{AA}$ parameter $\mathbb{Z}$ can affect the resulting ranking. Specifically, the highest-ranked alternative changes from $\mathcal{B}_4$ at $\mathbb{Z} = 0.5$ to $\mathcal{B}_1$ at $\mathbb{Z} = 1, 2$, and 5 , and to $\mathcal{B}_3$ at $\mathbb{Z} = 10$. However, $\mathcal{B}_1$ remains among the first two alternatives across all five tested values of $\mathbb{Z}$. These findings demonstrate that the selection of the $\mathcal{AA}$ parameter can influence the final decision and should therefore be chosen carefully according to the decision-making context.

6.1 Monte Carlo Sensitivity Analysis of Criterion Weights

To further examine the variation in the proposed *MC_{GDM}*-TOPSIS results with respect to the criterion weights, a Monte Carlo sensitivity analysis was conducted. The analysis systematically generates a large number of criterion-weight combinations. Specifically, the criterion weights satisfy $w_j \geq 0$ and $\sum_{j=1}^5 w_j = 1$.

We generated 10,000 criterion-weight vectors from a Dirichlet distribution with parameters (1,1,1,1,1), which gives a uniform distribution over the five-dimensional weight simplex. Each generated weight vector was normalized numerically to ensure that its components satisfy the unit-sum constraint. The $\mathcal{AA}$ parameter was fixed at $\mathbb{Z} = 2$, while the criterion weights were varied across the simulations. The random seed was fixed at 2026 to ensure reproducibility. For each sampled weight vector, the complete weighting and modified *TOPSIS* procedure was repeated, and the closeness coefficients of the five alternatives were calculated. The frequency with which each alternative obtained the first rank was then recorded. The results are presented in Table 9.

Table 9

First-rank frequency obtained from 10,000 Monte Carlo criterion-weight simulations

Alternative	First-rank count	First-rank frequency (%)
$\mathcal{B}_1$	6540	65.40
$\mathcal{B}_2$	2303	23.03
$\mathcal{B}_3$	33	0.33
$\mathcal{B}_4$	1096	10.96
$\mathcal{B}_5$	28	0.28

Table 10

Average and median ranks across the Monte Carlo simulations

Alternative	Average rank	Median rank
$\mathcal{B}_1$	1.5045	1
$\mathcal{B}_2$	2.2550	2
$\mathcal{B}_3$	3.2440	3
$\mathcal{B}_4$	3.2180	4
$\mathcal{B}_5$	4.7785	5

Table 11

Spearman rank correlation between Monte Carlo scenarios and the baseline ranking

Statistic	Spearman correlation
Mean	0.7511
Median	0.9000
Minimum	-1.0000
Maximum	1.0000
Standard deviation	0.3791
Exact baseline ranking frequency (%)	37.05

The Monte Carlo results show how the alternatives' rankings vary as the criterion weights change, including the frequency with which each alternative attains first rank. As shown in Table 9, $\mathcal{B}_1$ obtained the first rank in 65.40% of the 10,000 simulations, followed by $\mathcal{B}_2$ with 23.03% and $\mathcal{B}_4$ with

10.96%. The first-rank frequencies of $\mathcal{B}_3$ and $\mathcal{B}_5$ were 0.33% and 0.28%, respectively. Thus, the Monte Carlo analysis examines criterion-weight variation across a wider range of configurations than the five manually selected scenarios.

Table 10 provides further information about ranking variation through the average and median ranks. The median ranks of $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, and $\mathcal{B}_5$ were 1, 2, 3, 4, and 5, respectively. The corresponding average ranks were 1.5045, 2.2550, 3.2440, 3.2180, and 4.7785.

Furthermore, the Spearman rank-correlation analysis was performed to examine the consistency of the rankings obtained under the Monte Carlo weight scenarios with respect to the baseline ranking. The results are summarized in Table 11. The mean Spearman correlation was 0.7511, with a median value of 0.9000, while the minimum and maximum correlations were -1.0000 and 1.0000 , respectively. The standard deviation was 0.3791. The exact baseline ranking was reproduced in 37.05% of the simulations. These results indicate that variations in the criterion weights can alter the relative ranking of the alternatives. Therefore, the proposed framework should not be interpreted as producing an invariant ranking for all possible criterion-weight configurations. Instead, the Monte Carlo analysis quantifies the extent to which the decision results vary over the feasible weight space.

The Monte Carlo analysis further indicates that the ranking varies under perturbations of the criterion weights. The exact baseline ranking is recovered in 37.05% of the simulations, while $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_4$ are ranked first in 65.40%, 23.03%, and 10.96% of the simulations, respectively. The mean Spearman rank correlation is 0.7511, although the minimum value reaches -1 . These results indicate that the ranking is sensitive to variations in the criterion weights.

7. Comparative Analysis with an Existing TpVFF Method

To further examine the performance of the proposed framework, a comparative analysis was conducted with an existing aggregation approach developed for TpVFF numbers. In particular, the Dombi weighted geometric operator for TpVFF numbers (TpVFFDWG) proposed by Kaviyarasu *et al.* [24] was considered. The selected method is directly compatible with the TpVFF structure used in the present study.

For the comparison, we used the same decision matrix, expert weights, criterion weights, and modified *TOPSIS* procedure for both approaches. The only change was the aggregation operator used to combine the expert assessments. The proposed approach employs the $\mathcal{AA}$ aggregation mechanism, whereas the benchmark approach employs the Dombi weighted geometric aggregation operator. The Dombi parameter was fixed at $\gamma = 2$, consistent with the numerical implementation presented above. This design isolates the effect of the aggregation mechanism while keeping the other decision-making conditions unchanged.

The resulting score matrix obtained using the Dombi-based aggregation procedure is presented in Table 12. The corresponding PIS and NIS are obtained from the maximum and minimum score values of each criterion, respectively.

Table 12
Score matrix obtained using the Dombi TpVFFDWG aggregation operator

Alternative	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$
$\mathcal{B}_1$	0.180898	-0.004290	0.047458	-0.009459	0.036166
$\mathcal{B}_2$	0.079569	0.047821	0.153501	-0.073825	-0.002808
$\mathcal{B}_3$	0.110692	0.060970	0.055172	-0.046686	0.040231
$\mathcal{B}_4$	0.023747	-0.024149	0.039902	0.070016	0.080362
$\mathcal{B}_5$	-0.050236	0.048063	-0.021292	0.032194	0.025192

The PIS and NIS obtained from the Dombi-based score matrix are given by

$$\text{PIS} = (0.180898, 0.060970, 0.153501, 0.070016, 0.080362),$$

and

$$\text{NIS} = (-0.050236, -0.024149, -0.021292, -0.073825, -0.002808).$$

Using these ideal solutions, the separation measures and closeness coefficients were calculated using the same modified *TOPSIS* procedure employed for the proposed *AA* framework. The results are presented in Table 13.

Table 13
Comparison of the proposed Aczel-Alsina method and Dombi TpVFFDWG method

Alternative	Aczel-Alsina			Dombi TpVFFDWG		
	$\mathcal{D}(\mathcal{V}_{\mathcal{T}_p}^+, \mathfrak{B}_j)$	$\mathcal{D}(\mathcal{N}_{\mathcal{T}_p}^-, \mathfrak{B}_j)$	$\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)$	$\mathcal{D}(\mathcal{V}_{\mathcal{T}_p}^+, \mathfrak{B}_j)$	$\mathcal{D}(\mathcal{N}_{\mathcal{T}_p}^-, \mathfrak{B}_j)$	$\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)$
$\mathcal{B}_1$	0.093608	0.156654	0.625963	0.154186	0.253389	0.621699
$\mathcal{B}_2$	0.120666	0.146975	0.549150	0.195059	0.229307	0.540352
$\mathcal{B}_3$	0.114741	0.112195	0.494390	0.172706	0.203909	0.541426
$\mathcal{B}_4$	0.129961	0.123032	0.486305	0.211770	0.191900	0.475388
$\mathcal{B}_5$	0.178185	0.096379	0.351025	0.297685	0.131296	0.306065

The ranking obtained from the proposed *AA* approach is $\mathcal{B}_1 > \mathcal{B}_2 > \mathcal{B}_3 > \mathcal{B}_4 > \mathcal{B}_5$. In contrast, the Dombi TpVFFDWG approach produces the ranking $\mathcal{B}_1 > \mathcal{B}_3 > \mathcal{B}_2 > \mathcal{B}_4 > \mathcal{B}_5$. Thus, both approaches rank $\mathcal{B}_1$ first and $\mathcal{B}_5$ last in the considered numerical illustration. The only difference in the relative ordering occurs between $\mathcal{B}_2$ and $\mathcal{B}_3$. Under the proposed *AA* approach, $\mathcal{B}_2$ is ranked above $\mathcal{B}_3$, whereas the Dombi TpVFFDWG approach places $\mathcal{B}_3$ above $\mathcal{B}_2$. This difference indicates that the choice of aggregation mechanism can influence the relative discrimination among intermediate alternatives. The discrimination power was calculated as the difference between the maximum and minimum closeness coefficients.

For the proposed *AA* approach, the discrimination power is

$$DP_{AA} = \max(\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)) - \min(\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)) = 0.625963 - 0.351025 = 0.274938.$$

For the Dombi TpVFFDWG approach, the discrimination power is

$$DP_{\text{Dombi}} = \max(\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)) - \min(\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)) = 0.621699 - 0.306065 = 0.315634.$$

Table 14 summarizes the discrimination-power values and the rank-correlation coefficient for the two approaches. The Spearman rank correlation coefficient of $\rho = 0.900000$ indicates a strong posi-

Table 14
Discrimination power and rank correlation of the comparative methods

Measure	Aczel-Alsina	Dombi TpVFFDWG
Maximum $\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)$	0.625963	0.621699
Minimum $\mathcal{CC}_{\mathcal{T}_{p_i}}(\mathfrak{B}_j)$	0.351025	0.306065
Discrimination power	0.274938	0.315634
Rank correlation		
Spearman's ρ	0.900000	

tive association between the two rankings. The rankings are similar in the considered numerical illustration, although the second- and third-ranked alternatives are interchanged.

The comparative analysis shows that the rankings produced by the two aggregation approaches are similar in the considered numerical illustration. At the same time, the difference observed between $\mathcal{B}_2$ and $\mathcal{B}_3$ demonstrates that the aggregation mechanism can affect the relative ranking of alternatives with closer performance levels. The discrimination power also differs between the two approaches, with values of 0.274938 and 0.315634 for the *AA* and Dombi approaches, respectively.

This comparison is intended to examine methodological consistency under a common decision-making setting rather than to establish the superiority of one aggregation operator over another. The two approaches employ different aggregation mechanisms and may therefore respond differently to the distribution of $\mathcal{M}$ and $\mathcal{NM}$ information. Consequently, the observed ranking differences reflect the sensitivity of the decision outcome to the selected aggregation mechanism.

7.1 Key Features of the Proposed Framework

- The proposed TpVFF $\mathcal{AA}$ framework provides a representation of uncertainty, vagueness, and hesitation by allowing a broader range of $\mathcal{M}$ and $\mathcal{NM}$ values under the Fermatean condition.
- The integration of $\mathcal{AA}$ $\bar{T}$ and $\bar{S}$ provides an aggregation mechanism for modelling nonlinear interactions among the input assessments.
- By incorporating weighted, ordered weighted, and hybrid geometric aggregation processes within the same framework, the proposed operators $TpVFFAAWG$, $TpVFFAAOWG$, and $TpVFFAAHG$ provide different aggregation mechanisms for decision-making applications.
- The proposed operators are supported by formal proofs of fundamental mathematical properties, including idempotency, monotonicity, and boundedness.
- For $MCGDM$, the proposed TpVFF-TOPSIS method provides a systematic procedure for evaluating and ranking alternatives under the TpVFF environment.
- The applicability of the proposed framework is illustrated through a disaster relief allocation problem.

8. Conclusion

In this study, we proposed several $\mathcal{AA}$ -based geometric aggregation operators for TpVFF numbers, namely, $TpVFFAAWG$, $TpVFFAAOWG$, and $TpVFFAAHG$, and established their properties through theorems concerning idempotency, boundedness, and monotonicity. Additionally, we developed a modified TpVFF-TOPSIS method using the proposed $\mathcal{AO}s$ and applied it to a hypothetical $MCGDM$ problem concerning disaster relief allocation. We also compared the proposed methodology with an existing Dombi-based approach to examine the consistency of the resulting rankings. Sensitivity and Monte Carlo analyses were conducted to examine how the decision results vary with the $\mathcal{AA}$ parameter and criterion weights. The results show that the ranking remains unchanged under some of the tested scenarios, while it varies under other parameter and criterion-weight configurations. Future research may consider applying the proposed methodology to other fuzzy environments and dynamic decision-making situations. Overall, the findings illustrate the applicability of the proposed framework as an aggregation-based decision-making approach for the considered TpVFF environment.

8.1 Limitation and Future Work

One limitation of this work is the computational complexity associated with the combination of multiple aggregation operators, $\mathcal{AA}$ operations, and TpVFF calculations. In addition, the selected $\mathcal{AA}$ parameter can influence the resulting decision outcomes. The proposed approach assumes that the criteria are independent; however, in many real-world decision-making situations, relationships may exist among the criteria that affect the final ranking outcomes. Future research may consider incorporating other aggregation operators, such as the Bonferroni mean, Maclaurin symmetric mean, Heronian mean, and power average operators. In particular, for emergency management and real-time decision-making systems, future research may consider dynamic decision-making models with time-varying criterion values. The proposed method may also be investigated in other application areas, such as risk assessment, financial investment, sustainable energy planning, supplier selection, and medical diagnosis. Future research may investigate hybrid models that combine TpVFF with other fuzzy environments, such as spherical fuzzy sets, T-spherical fuzzy sets, and q-rung orthopair fuzzy sets.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this study.

Acknowledgement

This research was not funded by any grant.

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