



Intelligent Decision Support Using Circular Bipolar Complex Neutrosophic Fuzzy Frank Power Aggregation

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ABSTRACT

Multi-attribute decision-making (MADM) problems often involve uncertain, bipolar, and complex information that cannot be adequately represented by conventional fuzzy set models. To address this limitation, this study introduces the Circular Bipolar Complex Neutrosophic Fuzzy Set (CBCNFS) framework and develops two aggregation operators: the Circular Bipolar Complex Neutrosophic Fuzzy Frank Power Averaging (CBCNFFPA) operator and the Circular Bipolar Complex Neutrosophic Fuzzy Frank Power Geometric (CBCNFFPG) operator. The proposed operators enable the aggregation of multidimensional uncertain information and satisfy fundamental properties, including idempotency, boundedness, and monotonicity. Based on these operators, a score-function-based MADM algorithm is developed to evaluate and rank alternatives. A numerical example involving hypothetical electric vehicle charging stations illustrates the application of the proposed framework to information characterised by bipolarity, complex-valued representations, neutrosophic uncertainty, and circular structures. Sensitivity analyses are conducted to examine the effects of variations in the Frank parameter, criterion weights, and CBCNF evaluations on the resulting rankings. The results illustrate the applicability of the proposed aggregation operators and their responses to changes in the considered parameters and evaluation conditions. The proposed framework provides an integrated approach to representing and aggregating multidimensional uncertain information in MADM problems.

1. Introduction

MADM problems frequently involve imprecise, incomplete, and conflicting information. In many practical decision environments, a precise numerical assessment may not adequately represent the

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opinions of decision makers. Zadeh [1] introduced fuzzy set (FS) theory to handle imprecise and vague information by permitting partial membership (M). By adding both M and NM degrees, Atanassov [2] expanded the idea to intuitionistic FS (IFS), offering a more comprehensive framework for representing uncertainty. By proposing neutrosophic logic to independently model truth (T), indeterminacy (I), and falsity (F), Smarandache [3] accomplished further generalization and greatly increased the flexibility of uncertainty representation. Frank aggregation operators (FAOs) were introduced by Frank [4,5], who established the basic associative properties that later became important for developing adaptable aggregation tools. Ramot *et al.* [6] proposed complex FSs (CFSs) with amplitude and phase terms to better characterize periodic or oscillatory uncertainty, marking the expansion of fuzzy theory into complex domains. Alkouri and Salleh [7] expanded this approach by combining complex numbers with IFS to enable more expressive decision frameworks. To increase flexibility in MCDM situations, Ali *et al.* [8] created complex T-spherical fuzzy FAOs, combining spherical fuzzy logic with Frank operators.

1.1 Literature Review

The development of advanced fuzzy frameworks has therefore followed the need to represent different dimensions of uncertainty. Bipolar models are useful when both positive and negative evaluations need to be considered simultaneously. Complex-valued models provide an opportunity to represent the amplitude and phase of information, whereas circular models introduce an additional dimension through a radius representing the uncertainty range of an evaluation. These developments reflect the different aspects of information that may be uncertain and provide a basis for developing hybrid fuzzy models. Zhang [9] proposed the idea of bipolarity in fuzzy systems, which permits the simultaneous representation of positive and negative information. To represent two-dimensional uncertainty, Mahmood and Rehman [10] presented bipolar CFSs (BCFSs) that combine bipolarity with complex-valued membership functions. The emergence of circular fuzzy environments is another noteworthy development. Khan *et al.* [11], developed divergence measures for such sets to quantify differences between alternatives. To model uncertainty with an extra degree of freedom using circular structures, Atanassov [12] developed circular IFSs. This was further investigated by Chen [13], who proposed assessment techniques under circular IFSs for decision-making applications.

Recent studies have increasingly focused on the development of aggregation operators (AOs) for representing and combining different types of uncertain information in MCDM problems. To make it easier to combine complex-valued data, Bi *et al.* [14] presented arithmetic aggregation procedures for CFSs. To improve aggregation in bipolar environments, Wei *et al.* [15] presented bipolar fuzzy Hamacher AOs. Gul [16] expanded the VIKOR approach to handle compromise solutions in MCDM situations. Kaviyarasu [17] applied neutrosophic graph theory has been applied for the comparison of internet streaming services.

The integration of several sophisticated fuzzy concepts has led to the development of bipolar BCFS aggregation operators, which were introduced by Qiyas *et al.* [18] and shown to be useful in decision-making situations. Similarly, Mahmood *et al.* [19] investigated the properties of several AOs in bipolar complicated fuzzy situations. Additionally, practical and environmental applications have been studied. Kaviyarasu *et al.* [20] have proposed Dombi weighted geometric aggregation operators for trapezoidal-valued Fermatean fuzzy numbers and their applications in multi-attribute group decision-making problems. In another recent study by Kaviyarasu *et al.* [21], a symmetric TOPSIS method using trapezoidal-valued neutrosophic fuzzy Aczel-Alsina operators has been proposed for AI-based hiring evaluation. Alkouri and Alshboul [22] used BCFSs to solve environmental impact assessment issues. Similarly, Ali and Yang [23] used circular bipolar BCFSs linguistic information to study renewable energy systems, while Rehman *et al.* [24] used BCFS in MADM approaches for selecting AI models. To address increasingly complex decision-making problems under uncertainty, these studies collectively demonstrate a progressive transition from simple fuzzy sets to highly integrated hybrid models that combine circularity, bipolarity, complex-valued information, and advanced AOs.

The above studies illustrate the evolution of fuzzy decision-making frameworks and aggregation

operators in representing various types of uncertainty. However, when bipolar, complex, neutrosophic, and circular information coexist in the same decision-making problem, there is a need for a unified approach to represent these different types of uncertainty. This need for integration motivates the present study, in which a framework of circular bipolar complex neutrosophic fuzzy sets (CBCNFSs) is developed, along with Frank power aggregation operators (FPAOs), for solving MADM problems.

1.2 Research Gap

In practical decision-making problems, evaluations may take into account positive and negative preferences, T, I, and F, complex-valued information, and uncertainty about the spread of an evaluation. While bipolar fuzzy models, complex fuzzy models, neutrosophic models, and circular fuzzy structures have individually addressed some of these requirements, a unified framework capable of representing these multiple dimensions simultaneously is still limited. In addition, in the case of conventional aggregation operators, weights are usually known and may not depend on the level of agreement between the evaluations under aggregation. This calls for an aggregation process that depends on the level of similarity between the given evaluations. This leads to a research gap in the development of a unified decision-making approach that would take into consideration circular, bipolar, complex, and neutrosophic information along with a similarity-based Power aggregation process. The intention of such a combination is not to introduce unnecessary mathematical complexity, but to provide a unified representation when these different forms of uncertainty genuinely occur simultaneously and cannot be adequately represented by a simpler model.

1.3 Motivation

Motivated by this gap, the current study proposes a CBCNFS approach to capture multidimensional and contradictory uncertainty. In this respect, the bipolar attribute allows for the simultaneous representation of positive and negative information, whereas the neutrosophic attribute represents truth, indeterminacy, and falsity. The complex valued attribute is used for capturing information with real and imaginary components, and the circular attribute incorporates an additional dimension of uncertainty by using the radius parameter. Based on this rich information structure, FPAOs are proposed for aggregating CBCNF evaluations. The Power approach does not assume any statistical, causal, or preference-based interrelationship among criteria; rather, it dynamically considers the relative importance of the aggregated evaluations based on their mutual resemblance. The Frank operations further provide a flexible environment for aggregation. Therefore, the proposed approach is particularly suitable for decision-making applications where such types of multiple uncertainties truly exist. Accordingly, the proposed framework should be selected according to the nature and complexity of the uncertainty involved in the particular decision-making problem.

1.4 Contribution of the Paper

The primary contributions of this study are summarized as follows:

- A unique CBCNFS framework is presented that combines complex-valued circular characteristics with positive, negative, and indeterminate information to efficiently describe multidimensional uncertainty.
- The CBCNFS model is utilized to generate new FPAOs that provide a parameterized and flexible approach for combining decision-related information, dynamically adjusting the influence of the aggregated evaluations based on their mutual support.
- To guarantee mathematical validity and consistency, the basic characteristics of the proposed aggregation operators, such as idempotency, boundedness, and monotonicity, are established.
- Based on the proposed CBCNFS Frank Power Aggregation framework, an MADM model is developed to rank alternatives under complex and uncertain decision-making environments.
- An illustrative application to electric vehicle charging station selection, considering several competing factors such as distance, waiting time, charging speed, cost, and dependability, demonstrates the usefulness of the proposed method.

1.5 Organisation of the Paper

The background information on neutrosophic sets, bipolar structures, complex fuzzy concepts, and FPAOs is reviewed in Section 2. The proposed CBCNFS framework, including its basic definitions and operating rules, is presented in Section 3. The FPAOs in the CBCNFS environment are developed and their key characteristics are examined in Section 4. A step-by-step decision-making algorithm and the proposed MADM approach are presented in Section 5. The EV charging station selection problem, numerical example, and detailed analysis are also presented to demonstrate the applicability of the proposed approach. In Section 6, a comparison is presented between the proposed operators and selected existing fuzzy aggregation approaches to discuss their structural characteristics and parameter settings.

2. Preliminaries

This section presents the fundamental concepts required for developing the proposed aggregation operators. The relevant aggregation operators are constructed using the *FTCN* and *FTN*.

Definition 1. [4, 5] For any two positive numbers $\mathfrak{D}_1, \mathfrak{D}_2 \in [0, 1]$, the *FTN* and *FTCN* are defined, respectively, as follows:

$$FTN^\xi(\mathfrak{D}_1, \mathfrak{D}_2) = \log_\xi \left(1 + \frac{(\xi^{\mathfrak{D}_1} - 1)(\xi^{\mathfrak{D}_2} - 1)}{(\xi - 1)} \right)$$

$$FTCN^\xi(\mathfrak{D}_1, \mathfrak{D}_2) = 1 - \log_\xi \left(1 + \frac{(\xi^{1-\mathfrak{D}_1} - 1)(\xi^{1-\mathfrak{D}_2} - 1)}{(\xi - 1)} \right)$$

where $\xi \geq 2$.

Definition 2. [25, 26] The power aggregation operators *PG* and *PA* are defined for a collection of positive real numbers $\mathfrak{D}_1, \mathfrak{D}_2 \dots \mathfrak{D}_\varrho$, respectively, as follows:

$$PA(\mathfrak{D}_1, \mathfrak{D}_2 \dots \mathfrak{D}_\varrho) = \sum_{g=1}^{\varrho} \frac{(1 + \eta(\mathfrak{D}))}{\sum_{g=1}^{\varrho} (1 + \eta(\mathfrak{D}))} \mathfrak{D}$$

$$PG(\mathfrak{D}_1, \mathfrak{D}_2 \dots \mathfrak{D}_\varrho) = \prod_{g=1}^{\varrho} \frac{(1 + \eta(\mathfrak{D}))}{\sum_{g=1}^{\varrho} (1 + \eta(\mathfrak{D}))} \mathfrak{D}$$

with $\eta(\mathfrak{D}) = \sum_{i \neq g=1}^{\varrho} S(\mathfrak{D}_i, \mathfrak{D})$, where $S(\mathfrak{D}_i, \mathfrak{D}) = 1 - D(\mathfrak{D}_i, \mathfrak{D})$ with some conditions,

- (1) $S(\mathfrak{D}_i, \mathfrak{D}) = S(\mathfrak{D}, \mathfrak{D}_i)$
- (2) $S(\mathfrak{D}_i, \mathfrak{D}) \in [0, 1]$
- (3) $S(\mathfrak{D}_i, \mathfrak{D}) > S(\mathfrak{D}_k, \mathfrak{D}_l)$, then $D(\mathfrak{D}_i, \mathfrak{D}) < D(\mathfrak{D}_k, \mathfrak{D}_l)$

Definition 3. [1] Let $\mathbb{A}$ be a universe of discourse. A fuzzy set (FS) A is described by a collection of ordered pairs as follows:

$$A = \{ \langle z, \mathbb{M}_{\mathbb{S}_N}(z) \rangle \mid z \in \mathbb{A} \},$$

where $\mathbb{M}_{\mathbb{S}_N}$ denotes the membership function of z in A and $\mathbb{M}_{\mathbb{S}_N} : \mathbb{A} \rightarrow [0, 1]$.

Definition 4. [2] Let $\mathbb{A}$ be a universe of discourse. An IIFS A is characterized by two independent functions as follows:

$$A = \{ \langle z, \mathbb{M}_{\mathbb{S}_N}(z), \mathbb{N}_{\mathbb{S}_N}(z) \rangle \mid z \in \mathbb{A} \},$$

where $\mathbb{M}_{\mathbb{S}_N}$ and $\mathbb{N}_{\mathbb{S}_N}$ denote the membership and non-membership degrees of $z \in \mathbb{A}$, respectively. These functions satisfy the following conditions: $\mathbb{M}_{\mathbb{S}_N}(z) : \mathbb{A} \rightarrow [0, 1]$, $\mathbb{N}_{\mathbb{S}_N}(z) : \mathbb{A} \rightarrow [0, 1]$, and $0 \leq \mathbb{M}_{\mathbb{S}_N}(z) + \mathbb{N}_{\mathbb{S}_N}(z) \leq 1$.

Definition 5. [3] A NFS A is characterized by three independent functions as follows:

$$A = \{ \langle z, \mathbb{T}_{\mathbb{S}_N}(z), \mathbb{I}_{\mathbb{S}_N}(z), \mathbb{F}_{\mathbb{S}_N}(z) \rangle \mid z \in \mathbb{A} \},$$

where $\mathbb{T}_{\mathbb{S}_N}$, $\mathbb{I}_{\mathbb{S}_N}$, and $\mathbb{F}_{\mathbb{S}_N}$ denote the truth ($\mathbb{T}$), indeterminacy ($\mathbb{I}$), and falsity ($\mathbb{F}$) degrees of $z \in \mathbb{A}$, respectively. These functions satisfy the following conditions: $\mathbb{T}_{\mathbb{S}_N}(z) : \mathbb{A} \rightarrow [0, 1]$, $\mathbb{I}_{\mathbb{S}_N}(z) : \mathbb{A} \rightarrow [0, 1]$, $\mathbb{F}_{\mathbb{S}_N}(z) : \mathbb{A} \rightarrow [0, 1]$, and $0 \leq \mathbb{T}_{\mathbb{S}_N}(z) + \mathbb{I}_{\mathbb{S}_N}(z) + \mathbb{F}_{\mathbb{S}_N}(z) \leq 3$.

Definition 6. [27] A complex neutrosophic set A , defined on a universe of discourse $\mathbb{A}$, is characterized by membership, indeterminacy, and non-membership components with complex-valued grades. For any $z \in \mathbb{A}$, it is represented as follows:

$$\mathbb{T}_{\mathbb{S}_N}(z) = P_{\mathbb{S}_N}(z)e^{iM_{\mathbb{S}_N}(z)}, \mathbb{I}_{\mathbb{S}_N}(z) = Q_{\mathbb{S}_N}(z)e^{iN_{\mathbb{S}_N}(z)}, \mathbb{F}_{\mathbb{S}_N}(z) = R_{\mathbb{S}_N}(z)e^{iO_{\mathbb{S}_N}(z)}$$

where $P_{\mathbb{S}_N}(z)$, $Q_{\mathbb{S}_N}(z)$, $R_{\mathbb{S}_N}(z)$ and $M_{\mathbb{S}_N}(z)$, $N_{\mathbb{S}_N}(z)$, $O_{\mathbb{S}_N}(z)$ are real-valued functions and $P_{\mathbb{S}_N}(z), Q_{\mathbb{S}_N}(z), R_{\mathbb{S}_N}(z) \in [0, 1]$ such that $0 \leq P_{\mathbb{S}_N}(z) + Q_{\mathbb{S}_N}(z) + R_{\mathbb{S}_N}(z) \leq 3$.

3. Circular Bipolar Complex Neutrosophic Fuzzy Sets

The growing complexity of modern decision-making issues calls for models that can account for various types of uncertainty at the same time. However, existing fuzzy and neutrosophic approaches are mostly focused on the use of real-valued information and cannot reflect the simultaneous existence of positive and negative information along with its periodic or phase-based nature. In real-life situations such as intelligent transportation systems, energy generation management and the selection of electric vehicle charging stations, decision information may naturally exhibit uncertainty, bipolarity, and dynamic characteristics. Hence, there is a need for a more sophisticated representation approach. For these purposes, this paper presents the idea of a novel model called the CBCNFS. The presented model combines the strengths of the circular fuzzy structures, bipolar information description and complex neutrosophic sets theory. The use of the circular feature adds flexibility through the introduction of the radius parameter. Also, the combination with bipolar information allows accounting for positive and negative information simultaneously. Finally, the use of complex-valued $\mathbb{T}$, $\mathbb{I}$ and $\mathbb{F}$ functions reflects the periodical or phase-based nature of uncertainty. Accordingly, a number of operational laws are introduced to maintain the structural characteristics of CBCNFSs. These operations constitute the mathematical basis upon which the Frank power aggregation operators will be developed in the following section. In addition to generalizing a number of fuzzy models, this framework allows for the development of decision-making techniques in highly uncertain situations.

Definition 7. For any set $\mathbb{V}$, a CBCNFS $\mathfrak{Z}_\chi$ is defined as follows:

$$\mathfrak{Z}_\chi = \left\{ \left(\mathcal{T}_{\mathfrak{Z}_\chi}^+(\mathfrak{D}_o), \mathcal{T}_{\mathfrak{Z}_\chi}^-(\mathfrak{D}_o), \mathcal{I}_{\mathfrak{Z}_\chi}^+(\mathfrak{D}_o), \mathcal{I}_{\mathfrak{Z}_\chi}^-(\mathfrak{D}_o), \mathcal{F}_{\mathfrak{Z}_\chi}^+(\mathfrak{D}_o), \mathcal{F}_{\mathfrak{Z}_\chi}^-(\mathfrak{D}_o), R_{\mathfrak{Z}_\chi}(\mathfrak{D}_o) \right) \mid \mathfrak{D}_o \in \mathbb{V} \right\},$$

where $\mathcal{T}_{\mathfrak{Z}_\chi}^\pm(\mathfrak{D}_o) = \mathcal{T}_{\mathfrak{Z}_\chi}^{\Re\pm}(\mathfrak{D}_o) + i\mathcal{T}_{\mathfrak{Z}_\chi}^{\Im\pm}(\mathfrak{D}_o)$, $\mathcal{I}_{\mathfrak{Z}_\chi}^\pm(\mathfrak{D}_o) = \mathcal{I}_{\mathfrak{Z}_\chi}^{\Re\pm}(\mathfrak{D}_o) + i\mathcal{I}_{\mathfrak{Z}_\chi}^{\Im\pm}(\mathfrak{D}_o)$, $\mathcal{F}_{\mathfrak{Z}_\chi}^\pm(\mathfrak{D}_o) = \mathcal{F}_{\mathfrak{Z}_\chi}^{\Re\pm}(\mathfrak{D}_o) + i\mathcal{F}_{\mathfrak{Z}_\chi}^{\Im\pm}(\mathfrak{D}_o)$, $R_{\mathfrak{Z}_\chi}(\mathfrak{D}_o) = R_{\mathfrak{Z}_\chi}^{\Re}(\mathfrak{D}_o) + iR_{\mathfrak{Z}_\chi}^{\Im}(\mathfrak{D}_o)$. Here, $\mathcal{T}^+$, $\mathcal{I}^+$, and $\mathcal{F}^+$ denote the positive complex $\mathbb{T}$, $\mathbb{I}$ and $\mathbb{F}$ components respectively, whereas $\mathcal{T}^-$, $\mathcal{I}^-$, and $\mathcal{F}^-$ denote the corresponding negative components. The complex quantity R represents the circular radius associated with the corresponding evaluation. A Circular Bipolar Complex Neutrosophic Fuzzy Number (CBCNFN) is denoted by

$$\mathfrak{Z}_{\chi_g} = (\mathcal{T}_g^+, \mathcal{T}_g^-, \mathcal{I}_g^+, \mathcal{I}_g^-, \mathcal{F}_g^+, \mathcal{F}_g^-, R_g), \quad g = 1, 2, \dots, \mathfrak{L}.$$

The real and imaginary parts of the positive components and the circular radius satisfy

$$\mathcal{T}_g^{\Re+}, \mathcal{T}_g^{\Im+}, \mathcal{I}_g^{\Re+}, \mathcal{I}_g^{\Im+}, \mathcal{F}_g^{\Re+}, \mathcal{F}_g^{\Im+}, R_g^{\Re}, R_g^{\Im} \in [0, 1],$$

whereas the real and imaginary parts of the negative components satisfy

$$\mathcal{T}_g^{\Re-}, \mathcal{T}_g^{\Im-}, \mathcal{I}_g^{\Re-}, \mathcal{I}_g^{\Im-}, \mathcal{F}_g^{\Re-}, \mathcal{F}_g^{\Im-} \in [-1, 0].$$

Definition 8. For any two CBCNFNs $\mathfrak{Z}_1 = (\mathcal{T}_1^+, \mathcal{T}_1^-, \mathcal{I}_1^+, \mathcal{I}_1^-, \mathcal{F}_1^+, \mathcal{F}_1^-, R_1)$ and $\mathfrak{Z}_2 = (\mathcal{T}_2^+, \mathcal{T}_2^-, \mathcal{I}_2^+, \mathcal{I}_2^-, \mathcal{F}_2^+, \mathcal{F}_2^-, R_2)$, where each component is defined as in Definition 7, the following operational laws are defined.

$$(i) \mathfrak{Z}_{\chi_1} \oplus^{\text{tcn}} \mathfrak{Z}_{\chi_2} = \left(\begin{array}{l} \left(\mathcal{T}_1^{\Re+} + \mathcal{T}_2^{\Re+} - \mathcal{T}_1^{\Re+} \mathcal{T}_2^{\Re+} \right) + i \left(\mathcal{T}_1^{\Im+} + \mathcal{T}_2^{\Im+} - \mathcal{T}_1^{\Im+} \mathcal{T}_2^{\Im+} \right), - \left(\mathcal{T}_1^{\Re-} \mathcal{T}_2^{\Re-} \right) + i \left(- \left(\mathcal{T}_1^{\Im-} \mathcal{T}_2^{\Im-} \right) \right), \\ \left(\mathcal{I}_1^{\Re+} \mathcal{I}_2^{\Re+} \right) + i \left(\mathcal{I}_1^{\Im+} \mathcal{I}_2^{\Im+} \right), \left(\mathcal{I}_1^{\Re-} + \mathcal{I}_2^{\Re-} + \mathcal{I}_1^{\Re-} \mathcal{I}_2^{\Re-} \right) + i \left(\mathcal{I}_1^{\Im-} + \mathcal{I}_2^{\Im-} + \mathcal{I}_1^{\Im-} \mathcal{I}_2^{\Im-} \right), \\ \left(\mathcal{F}_1^{\Re+} \mathcal{F}_2^{\Re+} \right) + i \left(\mathcal{F}_1^{\Im+} \mathcal{F}_2^{\Im+} \right), \left(\mathcal{F}_1^{\Re-} + \mathcal{F}_2^{\Re-} + \mathcal{F}_1^{\Re-} \mathcal{F}_2^{\Re-} \right) + i \left(\mathcal{F}_1^{\Im-} + \mathcal{F}_2^{\Im-} + \mathcal{F}_1^{\Im-} \mathcal{F}_2^{\Im-} \right), \\ \left(\mathcal{R}_1^{\Re} + \mathcal{R}_2^{\Re} - \mathcal{R}_1^{\Re} \mathcal{R}_2^{\Re} \right) + i \left(\mathcal{R}_1^{\Im} + \mathcal{R}_2^{\Im} - \mathcal{R}_1^{\Im} \mathcal{R}_2^{\Im} \right) \end{array} \right)$$

$$(ii) \mathfrak{Z}_{\chi_1} \oplus^{\text{tn}} \mathfrak{Z}_{\chi_2} = \left(\begin{array}{l} \left(\mathcal{T}_1^{\Re+} + \mathcal{T}_2^{\Re+} - \mathcal{T}_1^{\Re} \mathcal{T}_2^{\Re+} \right) + i \left(\mathcal{T}_1^{\Im+} + \mathcal{T}_2^{\Im+} - \mathcal{T}_1^{\Im} \mathcal{T}_2^{\Im+} \right), - \left(\mathcal{T}_1^{\Re-} \mathcal{T}_2^{\Re-} \right) + i \left(- \left(\mathcal{T}_1^{\Im-} \mathcal{T}_2^{\Im-} \right) \right), \\ \left(\mathcal{I}_1^{\Re+} \mathcal{I}_2^{\Re+} \right) + i \left(\mathcal{I}_1^{\Im+} \mathcal{I}_2^{\Im+} \right), \left(\mathcal{I}_1^{\Re-} + \mathcal{I}_2^{\Re-} + \mathcal{I}_1^{\Re} \mathcal{I}_2^{\Re-} \right) + i \left(\mathcal{I}_1^{\Im-} + \mathcal{I}_2^{\Im-} + \mathcal{I}_1^{\Im} \mathcal{I}_2^{\Im-} \right), \\ \left(\mathcal{F}_1^{\Re+} \mathcal{F}_2^{\Re+} \right) + i \left(\mathcal{F}_1^{\Im+} \mathcal{F}_2^{\Im+} \right), \left(\mathcal{F}_1^{\Re-} + \mathcal{F}_2^{\Re-} + \mathcal{F}_1^{\Re} \mathcal{F}_2^{\Re-} \right) + i \left(\mathcal{F}_1^{\Im-} + \mathcal{F}_2^{\Im-} + \mathcal{F}_1^{\Im} \mathcal{F}_2^{\Im-} \right), \\ \left(R_1^{\Re} R_2^{\Re} \right) + i \left(R_1^{\Im} R_2^{\Im} \right) \end{array} \right)$$

$$(iii) \mathfrak{Z}_{\chi_1} \otimes^{\text{tcn}} \mathfrak{Z}_{\chi_2} = \left(\begin{array}{l} \left(\mathcal{I}_1^{\mathfrak{R}+} + \mathcal{I}_2^{\mathfrak{R}+} - \mathcal{I}_1^{\mathfrak{R}+} \mathcal{I}_2^{\mathfrak{R}+} \right) + i \left(\mathcal{I}_1^{\mathfrak{S}+} + \mathcal{I}_2^{\mathfrak{S}+} - \mathcal{I}_1^{\mathfrak{S}+} \mathcal{I}_2^{\mathfrak{S}+} \right), \left(\mathcal{I}_1^{\mathfrak{R}-} + \mathcal{I}_2^{\mathfrak{R}-} - \mathcal{I}_1^{\mathfrak{R}-} \mathcal{I}_2^{\mathfrak{R}-} \right) + i \left(\mathcal{I}_1^{\mathfrak{S}-} + \mathcal{I}_2^{\mathfrak{S}-} - \mathcal{I}_1^{\mathfrak{S}-} \mathcal{I}_2^{\mathfrak{S}-} \right), \\ \left(\mathcal{F}_1^{\mathfrak{R}+} + \mathcal{F}_2^{\mathfrak{R}+} - \mathcal{F}_1^{\mathfrak{R}+} \mathcal{F}_2^{\mathfrak{R}+} \right) + i \left(\mathcal{F}_1^{\mathfrak{S}+} + \mathcal{F}_2^{\mathfrak{S}+} - \mathcal{F}_1^{\mathfrak{S}+} \mathcal{F}_2^{\mathfrak{S}+} \right), \left(\mathcal{F}_1^{\mathfrak{R}-} + \mathcal{F}_2^{\mathfrak{R}-} - \mathcal{F}_1^{\mathfrak{R}-} \mathcal{F}_2^{\mathfrak{R}-} \right) + i \left(\mathcal{F}_1^{\mathfrak{S}-} + \mathcal{F}_2^{\mathfrak{S}-} - \mathcal{F}_1^{\mathfrak{S}-} \mathcal{F}_2^{\mathfrak{S}-} \right), \\ \left(R_1^{\mathfrak{R}} R_2^{\mathfrak{R}} \right) + i \left(R_1^{\mathfrak{S}} R_2^{\mathfrak{S}} \right) \end{array} \right)$$

$$(iv) \mathfrak{Z}_{\chi_1} \otimes^{\text{tn}} \mathfrak{Z}_{\chi_2} = \left(\begin{array}{l} \left(\mathcal{T}_1^{\Re+} + \mathcal{T}_2^{\Re+} \right) + i \left(\mathcal{T}_1^{\Im+} + \mathcal{T}_2^{\Im+} \right), \left(\mathcal{T}_1^{\Re-} + \mathcal{T}_2^{\Re-} + \mathcal{T}_1^{\Re-} \mathcal{T}_2^{\Re-} \right) + i \left(\mathcal{T}_1^{\Im-} + \mathcal{T}_2^{\Im-} + \mathcal{T}_1^{\Im-} \mathcal{T}_2^{\Im-} \right), \\ \left(\mathcal{I}_1^{\Re+} + \mathcal{I}_2^{\Re+} - \mathcal{I}_1^{\Re+} \mathcal{I}_2^{\Re+} \right) + i \left(\mathcal{I}_1^{\Im+} + \mathcal{I}_2^{\Im+} - \mathcal{I}_1^{\Im+} \mathcal{I}_2^{\Im+} \right), \left(-\mathcal{I}_1^{\Re-} \mathcal{I}_2^{\Re-} \right) + i \left(-\left(\mathcal{I}_1^{\Im-} \mathcal{I}_2^{\Im-} \right) \right), \\ \left(\mathcal{F}_1^{\Re+} + \mathcal{F}_2^{\Re+} - \mathcal{F}_1^{\Re+} \mathcal{F}_2^{\Re+} \right) + i \left(\mathcal{F}_1^{\Im+} + \mathcal{F}_2^{\Im+} - \mathcal{F}_1^{\Im+} \mathcal{F}_2^{\Im+} \right), \left(-\mathcal{F}_1^{\Re-} \mathcal{F}_2^{\Re-} \right) + i \left(-\left(\mathcal{F}_1^{\Im-} \mathcal{F}_2^{\Im-} \right) \right), \\ \left(\mathcal{R}_1^{\Re} + \mathcal{R}_2^{\Re} - \mathcal{R}_1^{\Re} \mathcal{R}_2^{\Re} \right) + i \left(\mathcal{R}_1^{\Im} + \mathcal{R}_2^{\Im} - \mathcal{R}_1^{\Im} \mathcal{R}_2^{\Im} \right) \end{array} \right)$$

$$(v) \mathfrak{Y}_{\text{tcn}} \mathfrak{Z}_{\chi_1} = \begin{pmatrix} \left(1 - (1 - \mathcal{T}_1^{\mathfrak{R}+})^{\mathfrak{Y}}\right) + i\left(1 - (1 - \mathcal{T}_1^{\mathfrak{S}+})^{\mathfrak{Y}}\right), -|\mathcal{T}_1^{\mathfrak{R}-}|^{\mathfrak{Y}} + i(-|\mathcal{T}_1^{\mathfrak{S}-}|^{\mathfrak{Y}}), \\ \left((\mathcal{I}_1^{\mathfrak{R}+})^{\mathfrak{Y}}\right) + i\left((\mathcal{I}_1^{\mathfrak{S}+})^{\mathfrak{Y}}\right), -1 + \left(1 + \mathcal{I}_1^{\mathfrak{R}-}\right)^{\mathfrak{Y}} + i\left(-1 + (1 + \mathcal{I}_1^{\mathfrak{S}-})^{\mathfrak{Y}}\right), \\ \left((\mathcal{F}_1^{\mathfrak{R}+})^{\mathfrak{Y}}\right) + i\left((\mathcal{F}_1^{\mathfrak{S}+})^{\mathfrak{Y}}\right), -1 + \left(1 + \mathcal{F}_1^{\mathfrak{R}-}\right)^{\mathfrak{Y}} + i\left(-1 + (1 + \mathcal{F}_1^{\mathfrak{S}-})^{\mathfrak{Y}}\right), \\ \left((R_1^{\mathfrak{R}})^{\mathfrak{Y}}\right), \left((R_1^{\mathfrak{S}})^{\mathfrak{Y}}\right) \end{pmatrix}$$

$$(vi) \dot{\mathfrak{Y}}_{\text{tn}\mathfrak{Z}_{\chi_1}} = \begin{pmatrix} \left(1 - (1 - \mathcal{T}_1^{\Re+})^{\dot{\mathfrak{Y}}}\right) + i\left(1 - (1 - \mathcal{T}_1^{\Im+})^{\dot{\mathfrak{Y}}}\right), -|\mathcal{T}_1^{\Re-}|^{\dot{\mathfrak{Y}}} + i(-|\mathcal{T}_1^{\Im-}|^{\dot{\mathfrak{Y}}}), \\ \left((\mathcal{I}_1^{\Re+})^{\dot{\mathfrak{Y}}}\right) + i\left((\mathcal{I}_1^{\Im+})^{\dot{\mathfrak{Y}}}\right), -1 + \left(1 + \mathcal{I}_1^{\Re-}\right)^{\dot{\mathfrak{Y}}} + i\left(-1 + (1 + \mathcal{I}_1^{\Im-})^{\dot{\mathfrak{Y}}}\right), \\ \left((\mathcal{F}_1^{\Re+})^{\dot{\mathfrak{Y}}}\right) + i\left((\mathcal{F}_1^{\Im+})^{\dot{\mathfrak{Y}}}\right), -1 + \left(1 + \mathcal{F}_1^{\Re-}\right)^{\dot{\mathfrak{Y}}} + i\left(-1 + (1 + \mathcal{F}_1^{\Im-})^{\dot{\mathfrak{Y}}}\right), \\ \left((R_1^{\Re})^{\dot{\mathfrak{Y}}}\right), \left((R_1^{\Im})^{\dot{\mathfrak{Y}}}\right) \end{pmatrix}$$

$$(vii) (\mathfrak{Z}_{\chi_1})^{\mathfrak{Y}_{\text{tn}}} = \begin{pmatrix} \left((\mathcal{T}_1^{\mathfrak{R}+})^{\mathfrak{Y}} + i((\mathcal{T}_1^{\mathfrak{S}+})^{\mathfrak{Y}}), -1 + (1 + \mathcal{T}_1^{\mathfrak{R}-})^{\mathfrak{Y}} + i(-1 + (1 + \mathcal{T}_1^{\mathfrak{S}-})^{\mathfrak{Y}}) \right), \\ \left(1 - (1 - \mathcal{I}_1^{\mathfrak{R}+})^{\mathfrak{Y}} + i(1 - (1 - \mathcal{I}_1^{\mathfrak{S}+})^{\mathfrak{Y}}), -|\mathcal{I}_1^{\mathfrak{R}-}|^{\mathfrak{Y}} + i(-|\mathcal{I}_1^{\mathfrak{S}-}|^{\mathfrak{Y}}) \right), \\ \left(1 - (1 - \mathcal{F}_1^{\mathfrak{R}+})^{\mathfrak{Y}} + i(1 - (1 - \mathcal{F}_1^{\mathfrak{S}+})^{\mathfrak{Y}}), -|\mathcal{F}_1^{\mathfrak{R}-}|^{\mathfrak{Y}} + i(-|\mathcal{F}_1^{\mathfrak{S}-}|^{\mathfrak{Y}}) \right), \\ \left(1 - (1 - R_1^{\mathfrak{R}})^{\mathfrak{Y}}, (1 - (1 - R_1^{\mathfrak{S}})^{\mathfrak{Y}}) \right) \end{pmatrix}$$

$$(viii) (\mathfrak{Z}_{\chi_1})^{\mathfrak{Y}_{\text{tn}}} = \begin{pmatrix} \left((\mathcal{T}_1^{\mathfrak{R}+})^{\mathfrak{Y}} + i((\mathcal{T}_1^{\mathfrak{S}+})^{\mathfrak{Y}}), -1 + (1 + \mathcal{T}_1^{\mathfrak{R}-})^{\mathfrak{Y}} + i(-1 + (1 + \mathcal{T}_1^{\mathfrak{S}-})^{\mathfrak{Y}}) \right), \\ \left(1 - (1 - \mathcal{I}_1^{\mathfrak{R}+})^{\mathfrak{Y}} + i(1 - (1 - \mathcal{I}_1^{\mathfrak{S}+})^{\mathfrak{Y}}), -|\mathcal{I}_1^{\mathfrak{R}-}|^{\mathfrak{Y}} + i(-|\mathcal{I}_1^{\mathfrak{S}-}|^{\mathfrak{Y}}) \right), \\ \left(1 - (1 - \mathcal{F}_1^{\mathfrak{R}+})^{\mathfrak{Y}} + i(1 - (1 - \mathcal{F}_1^{\mathfrak{S}+})^{\mathfrak{Y}}), -|\mathcal{F}_1^{\mathfrak{R}-}|^{\mathfrak{Y}} + i(-|\mathcal{F}_1^{\mathfrak{S}-}|^{\mathfrak{Y}}) \right), \\ \left((R_1^{\mathfrak{R}})^{\mathfrak{Y}}, (R_1^{\mathfrak{S}})^{\mathfrak{Y}} \right) \end{pmatrix}$$

3.1 Ranking Measures for CBCNFNs

Definition 9. Let $\mathfrak{Z}_1 = (\mathcal{T}_{\mathfrak{Z}_1}^+, \mathcal{T}_{\mathfrak{Z}_1}^-, \mathcal{I}_{\mathfrak{Z}_1}^+, \mathcal{I}_{\mathfrak{Z}_1}^-, \mathcal{F}_{\mathfrak{Z}_1}^+, \mathcal{F}_{\mathfrak{Z}_1}^-, R_{\mathfrak{Z}_1})$ and $\mathfrak{Z}_2 = (\mathcal{T}_{\mathfrak{Z}_2}^+, \mathcal{T}_{\mathfrak{Z}_2}^-, \mathcal{I}_{\mathfrak{Z}_2}^+, \mathcal{I}_{\mathfrak{Z}_2}^-, \mathcal{F}_{\mathfrak{Z}_2}^+, \mathcal{F}_{\mathfrak{Z}_2}^-, R_{\mathfrak{Z}_2})$ be two CBCNFNs. The partial order $\mathfrak{Z}_1 \preceq \mathfrak{Z}_2$ is defined by the following conditions: $\mathcal{T}_{\mathfrak{Z}_1}^{\mathfrak{R}+} \leq \mathcal{T}_{\mathfrak{Z}_2}^{\mathfrak{R}+}$, $\mathcal{T}_{\mathfrak{Z}_1}^{\mathfrak{S}+} \leq \mathcal{T}_{\mathfrak{Z}_2}^{\mathfrak{S}+}$, $\mathcal{I}_{\mathfrak{Z}_1}^{\mathfrak{R}+} \geq \mathcal{I}_{\mathfrak{Z}_2}^{\mathfrak{R}+}$, $\mathcal{I}_{\mathfrak{Z}_1}^{\mathfrak{S}+} \geq \mathcal{I}_{\mathfrak{Z}_2}^{\mathfrak{S}+}$, $\mathcal{F}_{\mathfrak{Z}_1}^{\mathfrak{R}+} \geq \mathcal{F}_{\mathfrak{Z}_2}^{\mathfrak{R}+}$, $\mathcal{F}_{\mathfrak{Z}_1}^{\mathfrak{S}+} \geq \mathcal{F}_{\mathfrak{Z}_2}^{\mathfrak{S}+}$, and $\mathcal{T}_{\mathfrak{Z}_1}^{\mathfrak{R}-} \leq \mathcal{T}_{\mathfrak{Z}_2}^{\mathfrak{R}-}$, $\mathcal{T}_{\mathfrak{Z}_1}^{\mathfrak{S}-} \leq \mathcal{T}_{\mathfrak{Z}_2}^{\mathfrak{S}-}$, $\mathcal{I}_{\mathfrak{Z}_1}^{\mathfrak{R}-} \leq \mathcal{I}_{\mathfrak{Z}_2}^{\mathfrak{R}-}$, $\mathcal{I}_{\mathfrak{Z}_1}^{\mathfrak{S}-} \leq \mathcal{I}_{\mathfrak{Z}_2}^{\mathfrak{S}-}$, $\mathcal{F}_{\mathfrak{Z}_1}^{\mathfrak{R}-} \leq \mathcal{F}_{\mathfrak{Z}_2}^{\mathfrak{R}-}$, $\mathcal{F}_{\mathfrak{Z}_1}^{\mathfrak{S}-} \leq \mathcal{F}_{\mathfrak{Z}_2}^{\mathfrak{S}-}$. Furthermore, since the circular-radius components represent the degree of circular uncertainty, a smaller radius is regarded as preferable. Therefore, the corresponding radius components satisfy

$$R_{\mathfrak{Z}_1}^{\mathfrak{R}} \geq R_{\mathfrak{Z}_2}^{\mathfrak{R}}, \quad R_{\mathfrak{Z}_1}^{\mathfrak{S}} \geq R_{\mathfrak{Z}_2}^{\mathfrak{S}}.$$

Thus, $\mathfrak{Z}_1 \preceq \mathfrak{Z}_2$ indicates that $\mathfrak{Z}_2$ has at least as favorable preference-related information and no greater circular uncertainty than $\mathfrak{Z}_1$.

Definition 10. For any CBCNFN $\mathfrak{Z}_g$, the score function is defined as

$$S(\mathfrak{Z}_g) = \frac{1}{14} \left(\mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{R}+} + \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{S}+} - \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{R}+} - \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{S}+} - \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{R}+} - \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{S}+} + \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{R}-} + \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{S}-} \right. \\ \left. + \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{R}-} + \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{S}-} + \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{R}-} + \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{S}-} - \mathcal{R}_{\mathfrak{Z}_g}^{\mathfrak{R}} - \mathcal{R}_{\mathfrak{Z}_g}^{\mathfrak{S}} \right).$$

Here, larger positive truth components increase the score, whereas larger positive indeterminacy and falsity components decrease the score. For the negative components, values closer to zero are considered preferable. Since a larger circular radius represents greater circular uncertainty, the radius components enter the score negatively; hence, a smaller circular radius contributes more favorably to the score.

Proposition 1. Let $\mathfrak{Z}_g$ be a CBCNFN. Then the score function defined in Definition 10 satisfies $-\frac{6}{7} \leq S(\mathfrak{Z}_g) \leq \frac{1}{7}$.

Proof. According to Definition 7, the six positive components satisfy $\mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{R}+}, \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{S}+}, \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{R}+}, \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{R}+}, \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{S}+} \in [0, 1]$. Hence, $0 \leq \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{R}+} + \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{S}+} \leq 2$, $0 \leq \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{R}+} + \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{S}+} \leq 2$, and $0 \leq \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{R}+} + \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{S}+} \leq 2$. Therefore, $-4 \leq \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{R}+} + \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{S}+} - \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{R}+} - \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{S}+} - \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{R}+} - \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{S}+} \leq 2$. Furthermore, the six negative components belong to $[-1, 0]$. Consequently, $-6 \leq \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{R}-} + \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{S}-} + \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{R}-} + \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{S}-} + \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{R}-} + \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{S}-} \leq 0$. Since $R_{\mathfrak{Z}_g}^{\mathfrak{R}}, R_{\mathfrak{Z}_g}^{\mathfrak{S}} \in [0, 1]$, we have $-2 \leq -\mathcal{R}_{\mathfrak{Z}_g}^{\mathfrak{R}} - \mathcal{R}_{\mathfrak{Z}_g}^{\mathfrak{S}} \leq 0$. Combining the above inequalities gives $-12 \leq 14S(\mathfrak{Z}_g) \leq 2$. Dividing by 14, we obtain $-\frac{6}{7} \leq S(\mathfrak{Z}_g) \leq \frac{1}{7}$. This completes the proof. $\square$

Definition 11. Let $\mathfrak{Z}_g$ be a CBCNFN. The accuracy function is defined as

$$A(\mathfrak{Z}_g) = \frac{1}{16} \left[10 + 2\mathcal{T}_{\mathfrak{Z}_g}^{\Re+} + 2\mathcal{T}_{\mathfrak{Z}_g}^{\Im+} - \mathcal{I}_{\mathfrak{Z}_g}^{\Re+} - \mathcal{I}_{\mathfrak{Z}_g}^{\Im+} - \mathcal{F}_{\mathfrak{Z}_g}^{\Re+} - \mathcal{F}_{\mathfrak{Z}_g}^{\Im+} \right. \\ \left. + 2\mathcal{T}_{\mathfrak{Z}_g}^{\Re-} + 2\mathcal{T}_{\mathfrak{Z}_g}^{\Im-} + \mathcal{I}_{\mathfrak{Z}_g}^{\Re-} + \mathcal{I}_{\mathfrak{Z}_g}^{\Im-} + \mathcal{F}_{\mathfrak{Z}_g}^{\Re-} + \mathcal{F}_{\mathfrak{Z}_g}^{\Im-} - \mathcal{R}_{\mathfrak{Z}_g}^{\Re} - \mathcal{R}_{\mathfrak{Z}_g}^{\Im} \right].$$

The circular-radius components are included with negative coefficients because smaller radius values represent lower circular uncertainty.

Definition 12. For any CBCNFN $\mathfrak{Z}_g$, the circular-radius uncertainty index is defined as

$$\rho(\mathfrak{Z}_g) = \frac{1}{2} \left(R_{\mathfrak{Z}_g}^{\Re} + R_{\mathfrak{Z}_g}^{\Im} \right).$$

A smaller value of $\rho(\mathfrak{Z}_g)$ represents a lower degree of circular uncertainty and is therefore regarded as preferable.

Definition 13. Let $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ be two CBCNFNs. Their ranking is determined according to the following rules:

1. If $S(\mathfrak{Z}_1) > S(\mathfrak{Z}_2)$, then $\mathfrak{Z}_1 \succ \mathfrak{Z}_2$.
2. If $S(\mathfrak{Z}_1) = S(\mathfrak{Z}_2)$, the accuracy values of $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are compared. If $A(\mathfrak{Z}_1) > A(\mathfrak{Z}_2)$, then $\mathfrak{Z}_1 \succ \mathfrak{Z}_2$.
3. If $S(\mathfrak{Z}_1) = S(\mathfrak{Z}_2)$ and $A(\mathfrak{Z}_1) = A(\mathfrak{Z}_2)$, then their circular-radius uncertainty indices are compared. If $\rho(\mathfrak{Z}_1) < \rho(\mathfrak{Z}_2)$, then $\mathfrak{Z}_1 \succ \mathfrak{Z}_2$. If $\rho(\mathfrak{Z}_1) > \rho(\mathfrak{Z}_2)$, then $\mathfrak{Z}_2 \succ \mathfrak{Z}_1$. If $\rho(\mathfrak{Z}_1) = \rho(\mathfrak{Z}_2)$, then $\mathfrak{Z}_1 \sim \mathfrak{Z}_2$.

Thus, the proposed ranking procedure first considers the overall preference information through the score function, then the accuracy function, and finally the degree of circular uncertainty through the circular-radius uncertainty index.

Proposition 2. Let $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ be two CBCNFNs such that $\mathfrak{Z}_1 \preceq \mathfrak{Z}_2$ as defined in Definition 9. Then $S(\mathfrak{Z}_1) \leq S(\mathfrak{Z}_2)$, $A(\mathfrak{Z}_1) \leq A(\mathfrak{Z}_2)$, and $\rho(\mathfrak{Z}_1) \geq \rho(\mathfrak{Z}_2)$. Hence, the revised ranking procedure is monotone with respect to the partial order defined in Definition 9.

Proof. From Proposition 3, if $\mathfrak{Z}_1 \preceq \mathfrak{Z}_2$, then $S(\mathfrak{Z}_1) \leq S(\mathfrak{Z}_2)$. Similarly, Proposition 4 establishes that $A(\mathfrak{Z}_1) \leq A(\mathfrak{Z}_2)$. Furthermore, according to Definition 9, $R_{\mathfrak{Z}_1}^{\Re} \geq R_{\mathfrak{Z}_2}^{\Re}$ and $R_{\mathfrak{Z}_1}^{\Im} \geq R_{\mathfrak{Z}_2}^{\Im}$. Therefore,

$$\begin{aligned} \rho(\mathfrak{Z}_1) &= \frac{1}{2} (R_{\mathfrak{Z}_1}^{\Re} + R_{\mathfrak{Z}_1}^{\Im}) \\ &\geq \frac{1}{2} (R_{\mathfrak{Z}_2}^{\Re} + R_{\mathfrak{Z}_2}^{\Im}) \\ &= \rho(\mathfrak{Z}_2). \end{aligned}$$

Thus, $\mathfrak{Z}_2$ has no greater circular uncertainty than $\mathfrak{Z}_1$. Since a smaller circular-radius uncertainty index is considered preferable, the radius component is consistent with the proposed partial order. Hence, the three ranking quantities $S(\mathfrak{Z})$, $A(\mathfrak{Z})$, and $\rho(\mathfrak{Z})$ are consistent with the revised partial order, and the proposed ranking procedure is monotone with respect to that order. $\square$

Proposition 3. Let $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ be two CBCNFNs such that $\mathfrak{Z}_1 \preceq \mathfrak{Z}_2$ according to Definition 9. Then $S(\mathfrak{Z}_1) \leq S(\mathfrak{Z}_2)$. Thus, the revised score function is monotone with respect to the partial order defined in Definition 9.

Proof. Since $\mathfrak{Z}_1 \preceq \mathfrak{Z}_2$, by Definition 9, the positive-pole truth components satisfy $\mathcal{T}_{\mathfrak{Z}_1}^{\Re+} \leq \mathcal{T}_{\mathfrak{Z}_2}^{\Re+}$, $\mathcal{T}_{\mathfrak{Z}_1}^{\Im+} \leq \mathcal{T}_{\mathfrak{Z}_2}^{\Im+}$, whereas the positive-pole indeterminacy and falsity components satisfy $\mathcal{I}_{\mathfrak{Z}_1}^{\Re+} \geq \mathcal{I}_{\mathfrak{Z}_2}^{\Re+}$, $\mathcal{I}_{\mathfrak{Z}_1}^{\Im+} \geq \mathcal{I}_{\mathfrak{Z}_2}^{\Im+}$, and $\mathcal{F}_{\mathfrak{Z}_1}^{\Re+} \geq \mathcal{F}_{\mathfrak{Z}_2}^{\Re+}$, $\mathcal{F}_{\mathfrak{Z}_1}^{\Im+} \geq \mathcal{F}_{\mathfrak{Z}_2}^{\Im+}$. For the negative-pole components, Definition 9 gives $\mathcal{T}_{\mathfrak{Z}_1}^{\Re-} \leq \mathcal{T}_{\mathfrak{Z}_2}^{\Re-}$, $\mathcal{T}_{\mathfrak{Z}_1}^{\Im-} \leq \mathcal{T}_{\mathfrak{Z}_2}^{\Im-}$, $\mathcal{I}_{\mathfrak{Z}_1}^{\Re-} \leq \mathcal{I}_{\mathfrak{Z}_2}^{\Re-}$, $\mathcal{I}_{\mathfrak{Z}_1}^{\Im-} \leq \mathcal{I}_{\mathfrak{Z}_2}^{\Im-}$, and $\mathcal{F}_{\mathfrak{Z}_1}^{\Re-} \leq \mathcal{F}_{\mathfrak{Z}_2}^{\Re-}$, $\mathcal{F}_{\mathfrak{Z}_1}^{\Im-} \leq \mathcal{F}_{\mathfrak{Z}_2}^{\Im-}$. Moreover, since the

circular radius represents the degree of circular uncertainty, a smaller radius is regarded as preferable. Hence, the partial order includes $R_{\mathfrak{Z}_1}^{\mathfrak{R}} \geq R_{\mathfrak{Z}_2}^{\mathfrak{R}}, R_{\mathfrak{Z}_1}^{\mathfrak{S}} \geq R_{\mathfrak{Z}_2}^{\mathfrak{S}}$. The revised score function is defined by

$$S(\mathfrak{Z}) = \frac{1}{14} \left(\mathcal{T}_3^{\mathfrak{R}+} + \mathcal{T}_3^{\mathfrak{S}+} - \mathcal{I}_3^{\mathfrak{R}+} - \mathcal{I}_3^{\mathfrak{S}+} - \mathcal{F}_3^{\mathfrak{R}+} - \mathcal{F}_3^{\mathfrak{S}+} + \mathcal{T}_3^{\mathfrak{R}-} + \mathcal{T}_3^{\mathfrak{S}-} \right. \\ \left. + \mathcal{I}_3^{\mathfrak{R}-} + \mathcal{I}_3^{\mathfrak{S}-} + \mathcal{F}_3^{\mathfrak{R}-} + \mathcal{F}_3^{\mathfrak{S}-} - R_{\mathfrak{Z}}^{\mathfrak{R}} - R_{\mathfrak{Z}}^{\mathfrak{S}} \right).$$

The positive-pole truth components occur with positive coefficients, so their ordering contributes in the direction $S(\mathfrak{Z}_1) \leq S(\mathfrak{Z}_2)$. The positive-pole indeterminacy and falsity components occur with negative coefficients, and their corresponding inequalities also contribute in the same direction. Similarly, the negative-pole components occur with positive coefficients, and their componentwise ordering is consistent with $S(\mathfrak{Z}_1) \leq S(\mathfrak{Z}_2)$. Finally, the radius components occur with negative coefficients in the revised score function. Since $R_{\mathfrak{Z}_1}^{\mathfrak{R}} \geq R_{\mathfrak{Z}_2}^{\mathfrak{R}}, R_{\mathfrak{Z}_1}^{\mathfrak{S}} \geq R_{\mathfrak{Z}_2}^{\mathfrak{S}}$, we have $-R_{\mathfrak{Z}_1}^{\mathfrak{R}} \leq -R_{\mathfrak{Z}_2}^{\mathfrak{R}}, -R_{\mathfrak{Z}_1}^{\mathfrak{S}} \leq -R_{\mathfrak{Z}_2}^{\mathfrak{S}}$. Therefore, the radius components also contribute consistently to $S(\mathfrak{Z}_1) \leq S(\mathfrak{Z}_2)$. Hence, every componentwise relation in Definition 9 contributes in the same direction, and consequently $S(\mathfrak{Z}_1) \leq S(\mathfrak{Z}_2)$. Thus, the revised score function is monotone with respect to the proposed partial order. $\square$

Proposition 4. Let $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ be two CBCNFNs such that $\mathfrak{Z}_1 \preceq \mathfrak{Z}_2$. Then $A(\mathfrak{Z}_1) \leq A(\mathfrak{Z}_2)$. Thus, the accuracy function is monotone with respect to the partial order defined in Definition 9.

Proof. From Definition 9, the positive-pole truth components of $\mathfrak{Z}_1$ are not greater than those of $\mathfrak{Z}_2$, while the positive-pole indeterminacy and falsity components of $\mathfrak{Z}_1$ are not smaller than those of $\mathfrak{Z}_2$. For the negative-pole components, the corresponding components of $\mathfrak{Z}_1$ are not greater than those of $\mathfrak{Z}_2$. Since the negative-pole components lie in $[-1, 0]$, the corresponding components of $\mathfrak{Z}_2$ are closer to zero and hence represent weaker unfavorable information. Furthermore, $R_{\mathfrak{Z}_1}^{\mathfrak{R}} \geq R_{\mathfrak{Z}_2}^{\mathfrak{R}}, R_{\mathfrak{Z}_1}^{\mathfrak{S}} \geq R_{\mathfrak{Z}_2}^{\mathfrak{S}}$, so $\mathfrak{Z}_1$ has no smaller circular uncertainty than $\mathfrak{Z}_2$. By the definition of the accuracy function, the positive-pole truth components have positive coefficients, whereas the positive-pole indeterminacy and falsity components have negative coefficients. The negative-pole components enter with positive coefficients, while the circular-radius components enter with negative coefficients. Hence, every componentwise relation in Definition 9 gives $A(\mathfrak{Z}_1) \leq A(\mathfrak{Z}_2)$. Therefore, the accuracy function is monotone with respect to the proposed partial order. $\square$

Remark 1. The equalities $S(\mathfrak{Z}_1) = S(\mathfrak{Z}_2)$ and $A(\mathfrak{Z}_1) = A(\mathfrak{Z}_2)$ do not necessarily imply $\mathfrak{Z}_1 = \mathfrak{Z}_2$. Indeed, CBCNFNs are multidimensional objects, whereas the score and accuracy functions map them to scalar values. Consequently, distinct CBCNFNs may nevertheless have identical score and accuracy values. However, the circular-radius components are explicitly incorporated into both ranking measures. Therefore, two CBCNFNs having identical truth, indeterminacy, and falsity components but different circular-radius components will generally yield different score and/or accuracy values, unless the corresponding differences are exactly offset by differences in the other components.

3.2 Proposed CBCNFFPA and CBCNFFPG Operators

Definition 14. For any two CBCNFNs $\mathfrak{Z}_g, g = 1, 2$, as defined in Definition 7, and for a Frank parameter $\alpha > 0, \alpha \neq 1$, the following operational laws are defined, where $\lg$ denotes the logarithm to base α .

For clarity, the above operations are shown component-wise in accordance with the positive, negative, and circular components of the CBCNFN. Although the resulting formulations are lengthy because of the multidimensional nature of the proposed model, they ensure that each component of the CBCNFN is aggregated appropriately.

$$\mathfrak{Z}_{\chi_1} \oplus^{\text{tn}} \mathfrak{Z}_{\chi_2} = \left(\begin{array}{l} \left(1 - \lg_a \left[1 + \frac{(a^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re+}} - 1)(a^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re+}} - 1)}{a-1} \right] \right) + i \left(1 - \lg_a \left[1 + \frac{(a^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Im+}} - 1)(a^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Im+}} - 1)}{a-1} \right] \right), \\ \left(-\lg_a \left[1 + \frac{(a^{-\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re-}} - 1)(a^{-\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re-}} - 1)}{a-1} \right] \right) + i \left(-\lg_a \left[1 + \frac{(a^{-\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Im-}} - 1)(a^{-\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Im-}} - 1)}{a-1} \right] \right), \\ \left(\lg_a \left[1 + \frac{(a^{\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re+}} - 1)(a^{\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re+}} - 1)}{a-1} \right] \right) + i \left(\lg_a \left[1 + \frac{(a^{\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Im+}} - 1)(a^{\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Im+}} - 1)}{a-1} \right] \right), \\ \left(-1 + \lg_a \left[1 + \frac{(a^{1+\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re-}} - 1)(a^{1+\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re-}} - 1)}{a-1} \right] \right) + i \left(-1 + \lg_a \left[1 + \frac{(a^{1+\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Im-}} - 1)(a^{1+\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Im-}} - 1)}{a-1} \right] \right), \\ \left(\lg_a \left[1 + \frac{(a^{\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Re+}} - 1)(a^{\mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Re+}} - 1)}{a-1} \right] \right) + i \left(\lg_a \left[1 + \frac{(a^{\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Im+}} - 1)(a^{\mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Im+}} - 1)}{a-1} \right] \right), \\ \left(-1 + \lg_a \left[1 + \frac{(a^{1+\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Re-}} - 1)(a^{1+\mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Re-}} - 1)}{a-1} \right] \right) + i \left(-1 + \lg_a \left[1 + \frac{(a^{1+\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Im-}} - 1)(a^{1+\mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Im-}} - 1)}{a-1} \right] \right), \\ \left(\lg_a \left[1 + \frac{(a^{R_{\mathfrak{Z}_{\chi_1}}^{\Re}} - 1)(a^{R_{\mathfrak{Z}_{\chi_2}}^{\Re}} - 1)}{a-1} \right] \right) + i \left(\lg_a \left[1 + \frac{(a^{R_{\mathfrak{Z}_{\chi_1}}^{\Im}} - 1)(a^{R_{\mathfrak{Z}_{\chi_2}}^{\Im}} - 1)}{a-1} \right] \right) \end{array} \right),$$

$$\mathfrak{Z}_{\chi_1} \otimes^{\text{tn}} \mathfrak{Z}_{\chi_2} = \left(\begin{array}{l} \left(\lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re+}} - 1 \right) \left(\mathfrak{a}^{\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re+}} - 1 \right)}{\mathfrak{a} - 1} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Im+}} - 1 \right) \left(\mathfrak{a}^{\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Im+}} - 1 \right)}{\mathfrak{a} - 1} \right] \right), \\ \left(-1 + \lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1+\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re-}} - 1 \right) \left(\mathfrak{a}^{1+\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re-}} - 1 \right)}{\mathfrak{a} - 1} \right] \right) + i \left(-1 + \lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1+\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Im-}} - 1 \right) \left(\mathfrak{a}^{1+\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Im-}} - 1 \right)}{\mathfrak{a} - 1} \right] \right), \\ \left(1 - \lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re+}} - 1 \right) \left(\mathfrak{a}^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re+}} - 1 \right)}{\mathfrak{a} - 1} \right] \right) + i \left(1 - \lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Im+}} - 1 \right) \left(\mathfrak{a}^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Im+}} - 1 \right)}{\mathfrak{a} - 1} \right] \right), \\ \left(-\lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re-}} - 1 \right) \left(\mathfrak{a}^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re-}} - 1 \right)}{\mathfrak{a} - 1} \right] \right) + i \left(-\lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Im-}} - 1 \right) \left(\mathfrak{a}^{1-\mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Im-}} - 1 \right)}{\mathfrak{a} - 1} \right] \right), \\ \left(1 - \lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1-\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Re+}} - 1 \right) \left(\mathfrak{a}^{1-\mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Re+}} - 1 \right)}{\mathfrak{a} - 1} \right] \right) + i \left(1 - \lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1-\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Im+}} - 1 \right) \left(\mathfrak{a}^{1-\mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Im+}} - 1 \right)}{\mathfrak{a} - 1} \right] \right), \\ \left(-\lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1-\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Re-}} - 1 \right) \left(\mathfrak{a}^{1-\mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Re-}} - 1 \right)}{\mathfrak{a} - 1} \right] \right) + i \left(-\lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{1-\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Im-}} - 1 \right) \left(\mathfrak{a}^{1-\mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Im-}} - 1 \right)}{\mathfrak{a} - 1} \right] \right), \\ \left(\lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{R_{\mathfrak{Z}_{\chi_1}}^{\Re}} - 1 \right) \left(\mathfrak{a}^{R_{\mathfrak{Z}_{\chi_2}}^{\Re}} - 1 \right)}{\mathfrak{a} - 1} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \frac{\left(\mathfrak{a}^{R_{\mathfrak{Z}_{\chi_1}}^{\Im}} - 1 \right) \left(\mathfrak{a}^{R_{\mathfrak{Z}_{\chi_2}}^{\Im}} - 1 \right)}{\mathfrak{a} - 1} \right] \right), \end{array} \right)$$

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Lemma 1. Let $\alpha > 0$ and $\alpha \neq 1$, and let $x, y \in [0, 1]$. Then the Frank t-norm and Frank t-conorm

defined by $\text{FTN}_a(x, y) = \log_a \left(1 + \frac{(a^x - 1)(a^y - 1)}{a - 1} \right)$ and $\text{FTCN}_a(x, y) = 1 - \log_a \left(1 + \frac{(a^{1-x} - 1)(a^{1-y} - 1)}{a - 1} \right)$ satisfy $\text{FTN}_a(x, y) \in [0, 1]$ and $\text{FTCN}_a(x, y) \in [0, 1]$. Hence, both the Frank t-norm and the Frank t-conorm are closed on $[0, 1]$.

Proof. Let $x, y \in [0, 1]$ and assume $a > 1$. Since the function $t \mapsto a^t$ is increasing, we have $a^0 = 1 \leq a^x \leq a$ and $a^0 = 1 \leq a^y \leq a$. Hence, $a^x - 1 \geq 0$ and $a^y - 1 \geq 0$, which implies that $1 + \frac{(a^x - 1)(a^y - 1)}{a - 1} \geq 1$. On the other hand, since $a^x - 1 \leq a - 1$ and $a^y - 1 \leq a - 1$, we obtain $(a^x - 1)(a^y - 1) \leq (a - 1)^2$. Since $a - 1 > 0$, it follows that $\frac{(a^x - 1)(a^y - 1)}{a - 1} \leq a - 1$, and therefore $1 + \frac{(a^x - 1)(a^y - 1)}{a - 1} \leq a$. Thus, $1 \leq 1 + \frac{(a^x - 1)(a^y - 1)}{a - 1} \leq a$. Since $\log_a \left(1 + \frac{(a^x - 1)(a^y - 1)}{a - 1} \right)$ is increasing for $a > 1$, we have $0 = \log_a(1) \leq \log_a \left(1 + \frac{(a^x - 1)(a^y - 1)}{a - 1} \right) \leq \log_a(a) = 1$.

Hence,

$$\text{FTN}_a(x, y) = \log_a \left(1 + \frac{(a^x - 1)(a^y - 1)}{a - 1} \right) \in [0, 1].$$

For the Frank t-conorm, note that $1 - x, 1 - y \in [0, 1]$ whenever $x, y \in [0, 1]$. Applying the above interval argument to $1 - x$ and $1 - y$, we obtain $\log_a \left(1 + \frac{(a^{1-x} - 1)(a^{1-y} - 1)}{a - 1} \right) \in [0, 1]$. Therefore, $\text{FTCN}_a(x, y) = 1 - \log_a \left(1 + \frac{(a^{1-x} - 1)(a^{1-y} - 1)}{a - 1} \right) \in [0, 1]$. Consequently, for every $x, y \in [0, 1]$, $\text{FTN}_a(x, y) \in [0, 1]$ and $\text{FTCN}_a(x, y) \in [0, 1]$. Thus, both the Frank t-norm and Frank t-conorm are closed on the unit interval. $\square$

3.3 Closure of the Proposed Operations and Aggregation Operators

The following results establish that all the proposed operations and aggregation operators preserve the admissible domain of a CBCNFN. In particular, it is proved component-wise that every positive real and imaginary component remains in $[0, 1]$, every negative real and imaginary component remains in $[-1, 0]$, and both radius components remain in $[0, 1]$.

Theorem 1. Let $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ be two valid CBCNFNs satisfying $\mathcal{T}^{\Re+}, \mathcal{T}^{\Im+}, \mathcal{I}^{\Re+}, \mathcal{I}^{\Im+}, \mathcal{F}^{\Re+}, \mathcal{F}^{\Im+} \in [0, 1]$, $\mathcal{T}^{\Re-}, \mathcal{T}^{\Im-}, \mathcal{I}^{\Re-}, \mathcal{I}^{\Im-}, \mathcal{F}^{\Re-}, \mathcal{F}^{\Im-} \in [-1, 0]$, and $R^{\Re}, R^{\Im} \in [0, 1]$. Then the operations $\oplus^{\text{tcn}}, \oplus^{\text{tn}}, \otimes^{\text{tcn}}, \otimes^{\text{tn}}$ together with the proposed scalar multiplication and exponentiation operations, are closed on the set of CBCNFNs.

Proof. **Closure of $\oplus^{\text{tcn}}$:**

We first establish the closure of $\oplus^{\text{tcn}}$ for positive $\mathcal{T}$ and radius components. By Definition 14, these components are obtained by applying the Frank t-conorm to two values $x, y \in [0, 1]$, given by

$$\text{FTCN}_a(x, y) = 1 - \log_a \left(1 + \frac{(a^{1-x} - 1)(a^{1-y} - 1)}{a - 1} \right).$$

Therefore, by Lemma 1, $\mathcal{T}^{\Re+}, \mathcal{T}^{\Im+}, R^{\Re}, R^{\Im} \in [0, 1]$. **Positive $\mathcal{I}$ and $\mathcal{F}$ components:** These components are obtained by applying the Frank t-norm,

$$\text{FTN}_a(x, y) = \log_a \left(1 + \frac{(a^x - 1)(a^y - 1)}{a - 1} \right), \quad x, y \in [0, 1].$$

Hence, by Lemma 1, $\mathcal{I}^{\Re+}, \mathcal{I}^{\Im+}, \mathcal{F}^{\Re+}, \mathcal{F}^{\Im+} \in [0, 1]$.

Negative $\mathcal{T}$ components: Let $x = -\mathcal{T}_1^{\Re-} \in [0, 1]$, $y = -\mathcal{T}_2^{\Re-} \in [0, 1]$, or, similarly, take the corresponding imaginary components. The resulting component is the negative of a Frank t-norm value. Since $\text{FTN}_a(x, y) \in [0, 1]$, we obtain $-\text{FTN}_a(x, y) \in [-1, 0]$. Therefore, $\mathcal{T}^{\Re-}, \mathcal{T}^{\Im-} \in [-1, 0]$.

Negative $\mathcal{I}$ and $\mathcal{F}$ components: For $x, y \in [-1, 0]$, we have $1 + x, 1 + y \in [0, 1]$. Therefore, $\text{FTN}_a(1 + x, 1 + y) \in [0, 1]$. Consequently, $-1 + \text{FTN}_a(1 + x, 1 + y) \in [-1, 0]$. Thus,

$\mathcal{I}^{\mathfrak{R}-}, \mathcal{I}^{\mathfrak{S}-}, \mathcal{F}^{\mathfrak{R}-}, \mathcal{F}^{\mathfrak{S}-} \in [-1, 0]$. Hence, $\mathfrak{Z}_1 \oplus^{\text{tcn}} \mathfrak{Z}_2$ is a valid CBCNFN.

Closure of $\oplus^{\text{tn}}$: The verification is identical, with the Frank t-norm and Frank t-conorm applied to the corresponding components according to Definition 14. The positive $\mathcal{T}$ components are obtained from the relevant Frank operation applied to values in $[0, 1]$, and therefore satisfy $\mathcal{T}^{\mathfrak{R}+}, \mathcal{T}^{\mathfrak{S}+} \in [0, 1]$. The negative $\mathcal{T}$ components are obtained by first transforming T^- to $1 + T^- \in [0, 1]$, applying the corresponding Frank t-norm, and then transforming the result back to $[-1, 0]$. Similarly, the positive $\mathcal{I}$ and $\mathcal{F}$ components are obtained by a Frank t-norm or Frank t-conorm applied to values in $[0, 1]$. Therefore, $\mathcal{I}^{\mathfrak{R}+}, \mathcal{I}^{\mathfrak{S}+}, \mathcal{F}^{\mathfrak{R}+}, \mathcal{F}^{\mathfrak{S}+} \in [0, 1]$. For the negative $\mathcal{I}$ and $\mathcal{F}$ components, the transformations involve arguments transformed into the interval $[0, 1]$. Consequently, $\mathcal{I}^{\mathfrak{R}-}, \mathcal{I}^{\mathfrak{S}-}, \mathcal{F}^{\mathfrak{R}-}, \mathcal{F}^{\mathfrak{S}-} \in [-1, 0]$. The radius components are obtained by applying Frank operations to values in $[0, 1]$. Hence, $R^{\mathfrak{R}}, R^{\mathfrak{S}} \in [0, 1]$. Therefore, $\mathfrak{Z}_1 \oplus^{\text{tn}} \mathfrak{Z}_2$ is a valid CBCNFN.

Closure of $\otimes^{\text{tcn}}$: For $\otimes^{\text{tcn}}$, the positive $\mathcal{T}$ components are Frank t-norms of values in $[0, 1]$. Hence, $\mathcal{T}^{\mathfrak{R}+}, \mathcal{T}^{\mathfrak{S}+} \in [0, 1]$. The negative $\mathcal{T}$ components are obtained from $1 + \mathcal{T}^- \in [0, 1]$ and therefore have the form $-1 + \text{FTN}_a(1 + \mathcal{T}_1^-, 1 + \mathcal{T}_2^-)$, which belongs to $[-1, 0]$. Thus, $\mathcal{T}^{\mathfrak{R}-}, \mathcal{T}^{\mathfrak{S}-} \in [-1, 0]$. The positive $\mathcal{I}$ and $\mathcal{F}$ components are Frank t-conorms of values in $[0, 1]$. Therefore, $\mathcal{I}^{\mathfrak{R}+}, \mathcal{I}^{\mathfrak{S}+}, \mathcal{F}^{\mathfrak{R}+}, \mathcal{F}^{\mathfrak{S}+} \in [0, 1]$. For the negative $\mathcal{I}$ and $\mathcal{F}$ components, the corresponding negative components are transformed into arguments in $[0, 1]$ and then processed by the appropriate Frank operation. Therefore, $\mathcal{I}^{\mathfrak{R}-}, \mathcal{I}^{\mathfrak{S}-}, \mathcal{F}^{\mathfrak{R}-}, \mathcal{F}^{\mathfrak{S}-} \in [-1, 0]$. The radius components are obtained from the corresponding Frank operation on $[0, 1]$. Hence, $R^{\mathfrak{R}}, R^{\mathfrak{S}} \in [0, 1]$. Therefore, $\mathfrak{Z}_1 \otimes^{\text{tcn}} \mathfrak{Z}_2$ is a valid CBCNFN.

Closure of $\otimes^{\text{tn}}$: The component-wise argument follows the same reasoning. The positive $\mathcal{T}$ components are Frank t-conorm values and therefore satisfy $\mathcal{T}^{\mathfrak{R}+}, \mathcal{T}^{\mathfrak{S}+} \in [0, 1]$. The negative $\mathcal{T}$ components are the negatives of Frank t-norm values applied to transformed arguments in $[0, 1]$. Hence, $\mathcal{T}^{\mathfrak{R}-}, \mathcal{T}^{\mathfrak{S}-} \in [-1, 0]$. The positive $\mathcal{I}$ and $\mathcal{F}$ components are Frank t-norm values. Therefore, $\mathcal{I}^{\mathfrak{R}+}, \mathcal{I}^{\mathfrak{S}+}, \mathcal{F}^{\mathfrak{R}+}, \mathcal{F}^{\mathfrak{S}+} \in [0, 1]$. The negative $\mathcal{I}$ and $\mathcal{F}$ components are obtained from transformed arguments in $[0, 1]$, followed by either a negative sign or a shift by -1 . Consequently, $\mathcal{I}^{\mathfrak{R}-}, \mathcal{I}^{\mathfrak{S}-}, \mathcal{F}^{\mathfrak{R}-}, \mathcal{F}^{\mathfrak{S}-} \in [-1, 0]$. Finally, the radius components are Frank t-norm or t-conorm values of arguments in $[0, 1]$, and hence $R^{\mathfrak{R}}, R^{\mathfrak{S}} \in [0, 1]$. Therefore, $\mathfrak{Z}_1 \otimes^{\text{tn}} \mathfrak{Z}_2$ is a valid CBCNFN.

Closure of scalar multiplication: Let $\mathfrak{J} > 0$. For every positive component $x \in [0, 1]$, the proposed scalar multiplication uses either $x^{\mathfrak{J}}$ or $1 - (1 - x)^{\mathfrak{J}}$. Since $0 \leq x^{\mathfrak{J}} \leq 1$ and $0 \leq 1 - (1 - x)^{\mathfrak{J}} \leq 1$, all positive components remain in $[0, 1]$. For every negative component $x \in [-1, 0]$, the scalar multiplication uses either $-1 + (1 + x)^{\mathfrak{J}}$ or $-|x|^{\mathfrak{J}}$. Since $1 + x \in [0, 1]$ and $|x| \in [0, 1]$, we have $0 \leq (1 + x)^{\mathfrak{J}} \leq 1$ and $0 \leq |x|^{\mathfrak{J}} \leq 1$. Therefore, $-1 \leq -1 + (1 + x)^{\mathfrak{J}} \leq 0$ and $-1 \leq -|x|^{\mathfrak{J}} \leq 0$. Hence, all negative components remain in $[-1, 0]$. The radius components have the form $R^{\mathfrak{J}}$ or $1 - (1 - R)^{\mathfrak{J}}$, where $R \in [0, 1]$. Therefore, $R^{\mathfrak{R}}, R^{\mathfrak{S}} \in [0, 1]$. Thus, scalar multiplication preserves the admissible CBCNFN domain.

Closure of exponentiation: Let $\mathfrak{J} > 0$. For every positive component $x \in [0, 1]$, the exponentiation operation produces either $x^{\mathfrak{J}} \in [0, 1]$ or $1 - (1 - x)^{\mathfrak{J}} \in [0, 1]$. For every negative component $x \in [-1, 0]$, the operation produces either $-1 + (1 + x)^{\mathfrak{J}} \in [-1, 0]$ or $-|x|^{\mathfrak{J}} \in [-1, 0]$. The radius components have the form $R^{\mathfrak{J}} \in [0, 1]$ or $1 - (1 - R)^{\mathfrak{J}} \in [0, 1]$. Therefore, exponentiation preserves the admissible CBCNFN domain. Consequently, all four binary operations, scalar multiplication, and exponentiation preserve the component-wise conditions $\mathcal{T}^{\mathfrak{R}+}, \mathcal{T}^{\mathfrak{S}+}, \mathcal{I}^{\mathfrak{R}+}, \mathcal{I}^{\mathfrak{S}+}, \mathcal{F}^{\mathfrak{R}+}, \mathcal{F}^{\mathfrak{S}+} \in [0, 1]$, $\mathcal{T}^{\mathfrak{R}-}, \mathcal{T}^{\mathfrak{S}-}, \mathcal{I}^{\mathfrak{R}-}, \mathcal{I}^{\mathfrak{S}-}, \mathcal{F}^{\mathfrak{R}-}, \mathcal{F}^{\mathfrak{S}-} \in [-1, 0]$, and $R^{\mathfrak{R}}, R^{\mathfrak{S}} \in [0, 1]$. Hence, all the proposed elementary operations are closed on the set of CBCNFNs. $\square$

4. Frank Power Aggregation operator

The efficiency of an MADM framework largely depends on its ability to properly aggregate uncertain information. Taking advantage of the flexibility of the Frank t-norm and Power aggregation operators, this study introduces new aggregation operators within the CBCNFS environment. It should be noted that the proposed Power aggregation operator does not directly represent any interaction among decision-making criteria from statistical, causal or preference perspectives. However, it dy-

namically adjusts the weight of each aggregated evaluation of CBCNF in accordance with their mutual support. Mutual support of evaluations is assessed in terms of their similarity, which is obtained via a distance function. Consequently, the proposed operators reveal the level of consensus of aggregated evaluations rather than interaction among criteria. The proposed CBCNFFPA, CBCNFFPA, CBCNFFPG and CBCNFFPWG operators are different from other recently proposed aggregation frameworks in terms of information richness and similarity-based Power weighting method

4.1 Closure of the Proposed Aggregation Operators

The closure of the elementary operations established in Theorem 1 is extended to the proposed CBCNFFPA, CBCNFFPA, CBCNFFPG, and CBCNFFPWG operators.

Lemma 2. For any collection of CBCNFNs $\mathfrak{Z}_1, \dots, \mathfrak{Z}_{\mathfrak{L}}$, the normalized Power weights ω_g satisfy $\omega_g \in (0, 1]$ for every $g = 1, \dots, \mathfrak{L}$ and $\sum_{g=1}^{\mathfrak{L}} \omega_g = 1$. The same conclusion holds for the weighted-power weights ϑ_g used in the CBCNFFPA and CBCNFFPWG operators.

Proof. For two admissible CBCNFNs, every absolute difference occurring in the distance function is bounded by 1, since corresponding positive components and radius components belong to $[0, 1]$, whereas corresponding negative components belong to $[-1, 0]$. Therefore, $0 \leq D(\mathfrak{Z}_i, \mathfrak{Z}_g) \leq 1$. Hence, $0 \leq S(\mathfrak{Z}_i, \mathfrak{Z}_g) = 1 - D(\mathfrak{Z}_i, \mathfrak{Z}_g) \leq 1$. Consequently, $0 \leq \mathcal{N}(\mathfrak{Z}_g) \leq \mathfrak{L} - 1$, and therefore $1 \leq 1 + \mathcal{N}(\mathfrak{Z}_g) \leq \mathfrak{L}$. Since every term in the numerator defining ω_g is positive and the denominator is the sum of all such positive terms, $0 < \omega_g \leq 1$. Moreover, by normalization, $\sum_{g=1}^{\mathfrak{L}} \omega_g = 1$. The same normalization argument applies to ϑ_g . Hence, $0 < \vartheta_g \leq 1$, $\sum_{g=1}^{\mathfrak{L}} \vartheta_g = 1$. $\square$

Lemma 3. Let $\mathfrak{Z}_g$ be a valid CBCNFN and let $\omega_g \in (0, 1]$. Then the scalar-multiplied CBCNFNs $\omega_{g\text{tcn}}\mathfrak{Z}_g, \omega_{g\text{tn}}\mathfrak{Z}_g$, and the exponentiated CBCNFNs $\mathfrak{Z}_g^{\omega_{g\text{tcn}}}, \mathfrak{Z}_g^{\omega_{g\text{tn}}}$, are valid CBCNFNs.

Proof. By Lemma 2, $\omega_g > 0$. Therefore, ω_g is an admissible scalar and exponent in the proposed operations. The scalar multiplication and exponentiation parts of Theorem 1 establish component-wise that all positive components remain in $[0, 1]$, all negative components remain in $[-1, 0]$, and both radius components remain in $[0, 1]$. Hence, all weighted and exponentiated elements appearing in the proposed aggregation operators are valid CBCNFNs. $\square$

Theorem 2. For any collection of admissible CBCNFNs $\mathfrak{Z}_1, \dots, \mathfrak{Z}_{\mathfrak{L}}$, the operators $CBCNFFPA^{\text{tcn}}(\mathfrak{Z}_1, \dots, \mathfrak{Z}_{\mathfrak{L}})$ and $CBCNFFPA^{\text{tn}}(\mathfrak{Z}_1, \dots, \mathfrak{Z}_{\mathfrak{L}})$ produce valid CBCNFNs for every $\mathfrak{L} \geq 1$.

Proof. We prove the result by induction on $\mathfrak{L}$. For $\mathfrak{L} = 1$, the normalized weight satisfies $\omega_1 = 1$. Therefore, $CBCNFFPA^{\text{tcn}}(\mathfrak{Z}_1) = 1 \odot \mathfrak{Z}_1 = \mathfrak{Z}_1$, which is a valid CBCNFN. For $\mathfrak{L} = 2$, Lemma 3 shows that $\omega_1 \odot \mathfrak{Z}_1$ and $\omega_2 \odot \mathfrak{Z}_2$ are valid CBCNFNs. By the closure of $\oplus^{\text{tcn}}$ proved in Theorem 1, their combination is also a valid CBCNFN. Assume now that $W_{\mathfrak{L}-1} = \bigoplus_{g=1}^{\text{tcn}\mathfrak{L}-1} \omega_g \odot \mathfrak{Z}_g$ is a valid CBCNFN. By Lemma 3, $\omega_{\mathfrak{L}} \odot \mathfrak{Z}_{\mathfrak{L}}$ is also a valid CBCNFN. Therefore, by Theorem 1, $W_{\mathfrak{L}-1} \oplus^{\text{tcn}} (\omega_{\mathfrak{L}} \odot \mathfrak{Z}_{\mathfrak{L}})$ is a valid CBCNFN. Hence, by induction, $CBCNFFPA^{\text{tcn}}$ is closed on the set of CBCNFNs for every $\mathfrak{L} \geq 1$. The proof for $CBCNFFPA^{\text{tn}}$ is identical, using the closure of $\oplus^{\text{tn}}$ established in Theorem 1. $\square$

Corollary 1. The operators $CBCNFFPA^{\text{tcn}}$ and $CBCNFFPA^{\text{tn}}$ produce valid CBCNFNs for every $\mathfrak{L} \geq 1$.

Proof. By Lemma 2, $\vartheta_g \in (0, 1]$ and $\sum_{g=1}^{\mathfrak{L}} \vartheta_g = 1$. Therefore, replacing ω_g by ϑ_g in the proof of Theorem 2, Lemma 3 guarantees that every weighted element is a valid CBCNFN, while Theorem 1 guarantees that every successive binary aggregation remains a valid CBCNFN. Hence, $CBCNFFPA^{\text{tcn}}$ and $CBCNFFPA^{\text{tn}}$ are closed on the set of CBCNFNs. $\square$

Theorem 3. For any collection of admissible CBCNFNs $\mathfrak{Z}_1, \dots, \mathfrak{Z}_{\mathfrak{L}}$, the operators $CBCNFFPG^{\text{tcn}}(\mathfrak{Z}_1, \dots, \mathfrak{Z}_{\mathfrak{L}})$ and $CBCNFFPG^{\text{tn}}(\mathfrak{Z}_1, \dots, \mathfrak{Z}_{\mathfrak{L}})$ produce valid CBCNFNs for every $\mathfrak{L} \geq 1$.

Proof. For $\mathfrak{L} = 1$, $\omega_1 = 1$, and hence $\mathfrak{Z}_1^{\omega_1} = \mathfrak{Z}_1$, which is a valid CBCNFN. For $\mathfrak{L} = 2$, Lemma 3 guarantees that $\mathfrak{Z}_1^{\omega_1}$ and $\mathfrak{Z}_2^{\omega_2}$ are valid CBCNFNs. By the closure of $\otimes^{\text{tcn}}$ proved in Theorem 1, their combination is a valid CBCNFN. Assume that $W_{\mathfrak{L}-1} = \bigotimes_{g=1}^{\text{tcn}\mathfrak{L}-1} \mathfrak{Z}_g^{\omega_g}$ is a valid CBCNFN. By Lemma 3, $\mathfrak{Z}_{\mathfrak{L}}^{\omega_{\mathfrak{L}}}$ is a valid CBCNFN. Therefore, by Theorem 1, $W_{\mathfrak{L}-1} \otimes^{\text{tcn}} \mathfrak{Z}_{\mathfrak{L}}^{\omega_{\mathfrak{L}}}$ is also a valid CBCNFN. Hence, $CBCNFFPG^{\text{tcn}}$ is closed on the set of CBCNFNs. The proof for $CBCNFFPG^{\text{tn}}$ is identical, using the closure of $\otimes^{\text{tn}}$ established in Theorem 1. $\square$

Corollary 2. The operators $CBCNFFPWG^{\text{tcn}}$ and $CBCNFFPWG^{\text{tn}}$ produce valid CBCNFNs for every $\mathfrak{L} \geq 1$.

Proof. By Lemma 2, $\vartheta_g \in (0, 1]$ for every g . Therefore, replacing ω_g by ϑ_g in Theorem 3, Lemma 3 ensures that every exponentiated element remains a valid CBCNFN, while Theorem 1 ensures that every successive $\otimes^{\text{tcn}}$ or $\otimes^{\text{tn}}$ combination remains within the admissible CBCNFN domain. Hence, $CBCNFFPWG^{\text{tcn}}$ and $CBCNFFPWG^{\text{tn}}$ are closed on the set of CBCNFNs. $\square$

Theorem 1 explicitly verifies the admissible intervals of every positive component, every negative component, and both radius components after each elementary operation. Lemmas 2 and 3, together with Theorems 2 and 3 and Corollary 1 and 2, extend this closure property to all four proposed Power aggregation operators. Therefore, the CBCNFFPA, CBCNFFPA, CBCNFFPG, and CBCNFFPWG operators produce valid CBCNFNs for every admissible collection of CBCNFN inputs.

4.2 Proposed CBCNFFPA and CBCNFFPG Operators

Definition 15. For any collection of CBCNFNs $\mathfrak{Z}_{\mathfrak{X}_g} = \left(\mathcal{T}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{R}+} + i\mathcal{T}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{S}+}, \mathcal{T}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{R}-} + i\mathcal{T}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{S}-}, \mathcal{I}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{R}+} + i\mathcal{I}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{S}+}, \mathcal{I}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{R}-} + i\mathcal{I}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{S}-}, \mathcal{F}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{R}+} + i\mathcal{F}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{R}-} + i\mathcal{F}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{S}-}, \mathcal{R}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{R}} + i\mathcal{R}_{\mathfrak{Z}_{\mathfrak{X}_g}}^{\mathfrak{S}} \right)_{g=1, 2, \dots, \mathfrak{L}}$, the CBCNFFPA operators are defined as follows:

$$CBCNFFPA^{\text{tn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \oplus_{g=1}^{\text{tn}\mathfrak{L}} \frac{(1 + \mathcal{N}(\mathfrak{Z}_g))}{\sum_{g=1}^{\mathfrak{L}} (1 + \mathcal{N}(\mathfrak{Z}_g))} \mathfrak{Z}_g$$

$$CBCNFFPA^{\text{tcn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \oplus_{g=1}^{\text{tcn}\mathfrak{L}} \frac{(1 + \mathcal{N}(\mathfrak{Z}_g))}{\sum_{g=1}^{\mathfrak{L}} (1 + \mathcal{N}(\mathfrak{Z}_g))} \mathfrak{Z}_g$$

Here, $CBCNFFPA^{\text{tn}}$ denotes the CBCNFFPA operator based on the $\mathcal{FTN}$, whereas $CBCNFFPA^{\text{tcn}}$ denotes the CBCNFFPA operator based on the $\mathcal{FTCN}$. $\mathcal{N}(\mathfrak{Z}_g) = \sum_{i \neq g=1}^{\mathfrak{L}} S(\mathfrak{Z}_i, \mathfrak{Z}_g)$, where $S(\mathfrak{Z}_i, \mathfrak{Z}_g) = 1 - D(\mathfrak{Z}_i, \mathfrak{Z}_g)$ satisfies the following conditions:

- (1) $S(\mathfrak{Z}_i, \mathfrak{Z}_g) = S(\mathfrak{Z}_g, \mathfrak{Z}_i)$
- (2) $S(\mathfrak{Z}_i, \mathfrak{Z}_g) \in [0, 1]$
- (3) If $S(\mathfrak{Z}_i, \mathfrak{Z}_g) > S(\mathfrak{Z}_k, \mathfrak{Z}_l)$, thus we have $D(\mathfrak{Z}_i, \mathfrak{Z}_g) < D(\mathfrak{Z}_k, \mathfrak{Z}_l)$

then we have

$$D(\mathfrak{Z}_i, \mathfrak{Z}_g) = \frac{1}{14} \left[\left| \mathcal{T}_{\mathfrak{Z}_i}^{\mathfrak{R}+} - \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{R}+} \right| + \left| \mathcal{T}_{\mathfrak{Z}_i}^{\mathfrak{S}+} - \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{S}+} \right| + \left| \mathcal{T}_{\mathfrak{Z}_i}^{\mathfrak{R}-} - \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{R}-} \right| + \left| \mathcal{T}_{\mathfrak{Z}_i}^{\mathfrak{S}-} - \mathcal{T}_{\mathfrak{Z}_g}^{\mathfrak{S}-} \right| \right. \\ \left. + \left| \mathcal{I}_{\mathfrak{Z}_i}^{\mathfrak{R}+} - \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{R}+} \right| + \left| \mathcal{I}_{\mathfrak{Z}_i}^{\mathfrak{S}+} - \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{S}+} \right| + \left| \mathcal{I}_{\mathfrak{Z}_i}^{\mathfrak{R}-} - \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{R}-} \right| + \left| \mathcal{I}_{\mathfrak{Z}_i}^{\mathfrak{S}-} - \mathcal{I}_{\mathfrak{Z}_g}^{\mathfrak{S}-} \right| \right. \\ \left. + \left| \mathcal{F}_{\mathfrak{Z}_i}^{\mathfrak{R}+} - \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{R}+} \right| + \left| \mathcal{F}_{\mathfrak{Z}_i}^{\mathfrak{S}+} - \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{S}+} \right| + \left| \mathcal{F}_{\mathfrak{Z}_i}^{\mathfrak{R}-} - \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{R}-} \right| + \left| \mathcal{F}_{\mathfrak{Z}_i}^{\mathfrak{S}-} - \mathcal{F}_{\mathfrak{Z}_g}^{\mathfrak{S}-} \right| \right. \\ \left. + \left| \mathcal{R}_{\mathfrak{Z}_i}^{\mathfrak{R}} - \mathcal{R}_{\mathfrak{Z}_g}^{\mathfrak{R}} \right| + \left| \mathcal{R}_{\mathfrak{Z}_i}^{\mathfrak{S}} - \mathcal{R}_{\mathfrak{Z}_g}^{\mathfrak{S}} \right| \right].$$

Property 1. For any two CBCNFNs $\mathfrak{Z}_i$ and $\mathfrak{Z}_g$, the proposed distance measure satisfies the following properties:

1. $D(\mathfrak{Z}_i, \mathfrak{Z}_g) \geq 0$.
2. $D(\mathfrak{Z}_i, \mathfrak{Z}_g) = D(\mathfrak{Z}_g, \mathfrak{Z}_i)$.
3. $D(\mathfrak{Z}_i, \mathfrak{Z}_g) = 0 \iff \mathfrak{Z}_i = \mathfrak{Z}_g$.

4. For any CBCNFNs $\mathfrak{Z}_i, \mathfrak{Z}_g$, and $\mathfrak{Z}_k$, $D(\mathfrak{Z}_i, \mathfrak{Z}_k) \leq D(\mathfrak{Z}_i, \mathfrak{Z}_g) + D(\mathfrak{Z}_g, \mathfrak{Z}_k)$.

Hence, D is a metric on the set of CBCNFNs.

Proof. The non-negativity follows directly from the fact that all the terms in the definition of D are absolute differences. Moreover, each component of a CBCNFN belongs to either $[0, 1]$ or $[-1, 0]$. Therefore, each absolute difference is bounded above by 1. Since the distance contains fourteen component-wise absolute differences,

$$0 \leq D(\mathfrak{Z}_i, \mathfrak{Z}_g) \leq \frac{14}{14} = 1.$$

The symmetry follows from $|x - y| = |y - x|$. Therefore, $D(\mathfrak{Z}_i, \mathfrak{Z}_g) = D(\mathfrak{Z}_g, \mathfrak{Z}_i)$. Now, suppose that $D(\mathfrak{Z}_i, \mathfrak{Z}_g) = 0$. Since the distance is the average of fourteen non-negative absolute differences, every term must be zero. Thus, $\mathcal{T}_{\mathfrak{Z}_i}^{\Re+} = \mathcal{T}_{\mathfrak{Z}_g}^{\Re+}, \mathcal{T}_{\mathfrak{Z}_i}^{\Im+} = \mathcal{T}_{\mathfrak{Z}_g}^{\Im+}, \mathcal{T}_{\mathfrak{Z}_i}^{\Re-} = \mathcal{T}_{\mathfrak{Z}_g}^{\Re-}, \mathcal{T}_{\mathfrak{Z}_i}^{\Im-} = \mathcal{T}_{\mathfrak{Z}_g}^{\Im-}, \mathcal{I}_{\mathfrak{Z}_i}^{\Re+} = \mathcal{I}_{\mathfrak{Z}_g}^{\Re+}, \mathcal{I}_{\mathfrak{Z}_i}^{\Im+} = \mathcal{I}_{\mathfrak{Z}_g}^{\Im+}, \mathcal{I}_{\mathfrak{Z}_i}^{\Re-} = \mathcal{I}_{\mathfrak{Z}_g}^{\Re-}, \mathcal{I}_{\mathfrak{Z}_i}^{\Im-} = \mathcal{I}_{\mathfrak{Z}_g}^{\Im-}, \mathcal{F}_{\mathfrak{Z}_i}^{\Re+} = \mathcal{F}_{\mathfrak{Z}_g}^{\Re+}, \mathcal{F}_{\mathfrak{Z}_i}^{\Im+} = \mathcal{F}_{\mathfrak{Z}_g}^{\Im+}, \mathcal{F}_{\mathfrak{Z}_i}^{\Re-} = \mathcal{F}_{\mathfrak{Z}_g}^{\Re-}, \mathcal{F}_{\mathfrak{Z}_i}^{\Im-} = \mathcal{F}_{\mathfrak{Z}_g}^{\Im-}, \mathcal{R}_{\mathfrak{Z}_i}^{\Re} = \mathcal{R}_{\mathfrak{Z}_g}^{\Re}, \mathcal{R}_{\mathfrak{Z}_i}^{\Im} = \mathcal{R}_{\mathfrak{Z}_g}^{\Im}$. Hence, $\mathfrak{Z}_i = \mathfrak{Z}_g$. Conversely, if $\mathfrak{Z}_i = \mathfrak{Z}_g$, then all fourteen absolute differences are zero and therefore $D(\mathfrak{Z}_i, \mathfrak{Z}_g) = 0$. Thus, $D(\mathfrak{Z}_i, \mathfrak{Z}_g) = 0 \iff \mathfrak{Z}_i = \mathfrak{Z}_g$. Finally, for any real numbers x, y , and z , $|x - z| \leq |x - y| + |y - z|$. Applying this inequality to each of the fourteen corresponding components and summing them gives

$$D(\mathfrak{Z}_i, \mathfrak{Z}_k) \leq D(\mathfrak{Z}_i, \mathfrak{Z}_g) + D(\mathfrak{Z}_g, \mathfrak{Z}_k).$$

Hence, D is a metric on the set of CBCNFNs. $\square$

Example 1. Consider three admissible CBCNF evaluations, represented in the order $(T^+, T^-, I^+, I^-, F^+, F^-, R)$, with identical real and imaginary parts for each component:

$$\mathfrak{Z}_{\chi_1} = (0.3 + i0.3, -0.6 - i0.6, 0.6 + i0.6, -0.5 - i0.5, 0.6 + i0.6, -0.5 - i0.5, 0.5 + i0.5),$$

$$\mathfrak{Z}_{\chi_2} = (0.1 + i0.1, -0.9 - i0.9, 0.8 + i0.8, -0.8 - i0.8, 0.8 + i0.8, -0.8 - i0.8, 0.1 + i0.1),$$

$$\mathfrak{Z}_{\chi_3} = (0.9 + i0.9, -0.1 - i0.1, 0.1 + i0.1, -0.1 - i0.1, 0.1 + i0.1, 0 + i0, 0.9 + i0.9).$$

All positive components and both radius components belong to $[0, 1]$, while all negative components belong to $[-1, 0]$. Hence, the three evaluations satisfy the admissibility conditions of Definition 7. For $\mathfrak{Z}_{\chi_1}$ and $\mathfrak{Z}_{\chi_2}$, the absolute differences between the corresponding CBCNF components are

$$\begin{aligned} |\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re+} - \mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re+}| &= 0.2, & |\mathcal{T}_{\mathfrak{Z}_{\chi_1}}^{\Re-} - \mathcal{T}_{\mathfrak{Z}_{\chi_2}}^{\Re-}| &= 0.3, \\ |\mathcal{I}_{\mathfrak{Z}_{\chi_1}}^{\Re+} - \mathcal{I}_{\mathfrak{Z}_{\chi_2}}^{\Re+}| &= 0.2, & |\mathcal{I}_{\mathfrak{Z}_{\chi_1}}^{\Re-} - \mathcal{I}_{\mathfrak{Z}_{\chi_2}}^{\Re-}| &= 0.3, \\ |\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Re+} - \mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Re+}| &= 0.2, & |\mathcal{F}_{\mathfrak{Z}_{\chi_1}}^{\Re-} - \mathcal{F}_{\mathfrak{Z}_{\chi_2}}^{\Re-}| &= 0.3. \end{aligned}$$

Since the real and imaginary parts have identical absolute differences, the sum of the twelve truth, indeterminacy, and falsity components is $2(0.2 + 0.3 + 0.2 + 0.3 + 0.2 + 0.3) = 3.0$. The radius contribution is $|\mathcal{R}_{\mathfrak{Z}_{\chi_1}}^{\Re} - \mathcal{R}_{\mathfrak{Z}_{\chi_2}}^{\Re}| + |\mathcal{R}_{\mathfrak{Z}_{\chi_1}}^{\Im} - \mathcal{R}_{\mathfrak{Z}_{\chi_2}}^{\Im}| = |0.5 - 0.1| + |0.5 - 0.1| = 0.8$.

Therefore,

$$D(\mathfrak{Z}_{\chi_1}, \mathfrak{Z}_{\chi_2}) = \frac{3.0 + 0.8}{14} = 0.271429.$$

Similarly, for $\mathfrak{Z}_{\chi_1}$ and $\mathfrak{Z}_{\chi_3}$, the sum of the twelve truth, indeterminacy, and falsity components is 6.0, and the radius contribution is $|\mathcal{R}_{\mathfrak{Z}_{\chi_1}}^{\Re} - \mathcal{R}_{\mathfrak{Z}_{\chi_3}}^{\Re}| + |\mathcal{R}_{\mathfrak{Z}_{\chi_1}}^{\Im} - \mathcal{R}_{\mathfrak{Z}_{\chi_3}}^{\Im}| = |0.5 - 0.9| + |0.5 - 0.9| = 0.8$. Hence,

$$D(\mathfrak{Z}_{\chi_1}, \mathfrak{Z}_{\chi_3}) = \frac{6.0 + 0.8}{14} = 0.485714.$$

For $\mathfrak{Z}_{\chi_2}$ and $\mathfrak{Z}_{\chi_3}$, the sum of the twelve truth, indeterminacy, and falsity components is 9.0, and

the radius contribution is

$$\left| \mathcal{R}_{\mathfrak{Z}_{\chi_2}}^{\mathfrak{R}} - \mathcal{R}_{\mathfrak{Z}_{\chi_3}}^{\mathfrak{R}} \right| + \left| \mathcal{R}_{\mathfrak{Z}_{\chi_2}}^{\mathfrak{S}} - \mathcal{R}_{\mathfrak{Z}_{\chi_3}}^{\mathfrak{S}} \right| = |0.1 - 0.9| + |0.1 - 0.9| = 1.6.$$

Thus, $D(\mathfrak{Z}_{\chi_2}, \mathfrak{Z}_{\chi_3}) = \frac{9.0 + 1.6}{14} = 0.757143$.

Therefore, the pairwise distances are

$$D(\mathfrak{Z}_{\chi_1}, \mathfrak{Z}_{\chi_2}) = 0.271429,$$

$$D(\mathfrak{Z}_{\chi_1}, \mathfrak{Z}_{\chi_3}) = 0.485714,$$

$$D(\mathfrak{Z}_{\chi_2}, \mathfrak{Z}_{\chi_3}) = 0.757143.$$

Using the support function $S(\mathfrak{Z}_{\chi_i}, \mathfrak{Z}_{\chi_g}) = 1 - D(\mathfrak{Z}_{\chi_i}, \mathfrak{Z}_{\chi_g})$, the corresponding support degrees are

$$S(\mathfrak{Z}_{\chi_1}, \mathfrak{Z}_{\chi_2}) = 0.728571,$$

$$S(\mathfrak{Z}_{\chi_1}, \mathfrak{Z}_{\chi_3}) = 0.514286,$$

$$S(\mathfrak{Z}_{\chi_2}, \mathfrak{Z}_{\chi_3}) = 0.242857.$$

Consequently,

$$\begin{aligned} N(\mathfrak{Z}_{\chi_1}) &= S(\mathfrak{Z}_{\chi_2}, \mathfrak{Z}_{\chi_1}) + S(\mathfrak{Z}_{\chi_3}, \mathfrak{Z}_{\chi_1}) \\ &= 0.728571 + 0.514286 \\ &= 1.242857, \end{aligned}$$

$$\begin{aligned} N(\mathfrak{Z}_{\chi_2}) &= S(\mathfrak{Z}_{\chi_1}, \mathfrak{Z}_{\chi_2}) + S(\mathfrak{Z}_{\chi_3}, \mathfrak{Z}_{\chi_2}) \\ &= 0.728571 + 0.242857 \\ &= 0.971429, \end{aligned}$$

and

$$\begin{aligned} N(\mathfrak{Z}_{\chi_3}) &= S(\mathfrak{Z}_{\chi_1}, \mathfrak{Z}_{\chi_3}) + S(\mathfrak{Z}_{\chi_2}, \mathfrak{Z}_{\chi_3}) \\ &= 0.514286 + 0.242857 \\ &= 0.757143. \end{aligned}$$

Therefore, according to the existing Power-weight definition,

$$\omega_g = \frac{1 + N(\mathfrak{Z}_{\chi_g})}{\sum_{j=1}^3 [1 + N(\mathfrak{Z}_{\chi_j})]}.$$

Since $\sum_{j=1}^3 [1 + N(\mathfrak{Z}_{\chi_j})] = 2.242857 + 1.971429 + 1.757143$ the normalized Power weights are

$$= 5.971429,$$

$$\begin{aligned} \omega_1 &= \frac{2.242857}{5.971429} = 0.375598, \\ \omega_2 &= \frac{1.971429}{5.971429} = 0.330144, \\ \omega_3 &= \frac{1.757143}{5.971429} = 0.294258. \end{aligned}$$

Clearly,

$$\omega_1 + \omega_2 + \omega_3 = 1.$$

Thus, the revised Power weights are

$$\omega_1 = 0.375598, \quad \omega_2 = 0.330144, \quad \omega_3 = 0.294258.$$

This example demonstrates that the distance function simultaneously considers the truth, indeterminacy, falsity, and circular-radius components of CBCNF evaluations. Together with Property 1, these

Theorem 4. For any collection of CBCNFNs, the CBCNFFPA operator produces a CBCNFN. That is $CBCNFFPA^{tcn}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_g) =$

$$\left(\begin{array}{l} \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-\mathcal{T}_{3\chi_g}^{\Re+}} - 1 \right)^{\omega_g} \right] \right) + i \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-\mathcal{T}_{3\chi_g}^{\Im+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{-\mathcal{T}_{3\chi_g}^{\Im-}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{-\mathcal{T}_{3\chi_g}^{\Re-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{\mathcal{T}_{3\chi_g}^{\Re+}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{\mathcal{T}_{3\chi_g}^{\Im+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-1 + \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left[a^{1+\mathcal{T}_{3\chi_g}^{\Re-}} - 1 \right]^{\omega_g} \right] \right) + i \left(-1 + \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1+\mathcal{T}_{3\chi_g}^{\Im-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{\mathcal{F}_{3\chi_g}^{\Re+}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{\mathcal{F}_{3\chi_g}^{\Im+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-1 + \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left[a^{1+\mathcal{F}_{3\chi_g}^{\Re-}} - 1 \right]^{\omega_g} \right] \right) + i \left(-1 + \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1+\mathcal{F}_{3\chi_g}^{\Im-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-R_{3\chi_g}^{\Re}} - 1 \right)^{\omega_g} \right] \right) + i \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-R_{3\chi_g}^{\Im}} - 1 \right)^{\omega_g} \right] \right), \end{array} \right) \quad (1)$$

$$CBCNFFPA^{\text{tn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) = \left(\begin{array}{l} \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 - \mathcal{T}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{R}+}} - 1 \right)^{\omega_g} \right] \right) + i \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 - \mathcal{T}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{S}+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{T}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{R}-}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{T}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{S}-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{I}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{R}+}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{I}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{S}+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 + \mathcal{I}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{R}-}} - 1 \right)^{\omega_g} \right] \right) + i \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 + \mathcal{I}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{S}-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{F}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{R}+}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{F}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{S}+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 + \mathcal{F}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{R}-}} - 1 \right)^{\omega_g} \right] \right) + i \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 + \mathcal{F}_{\mathfrak{z}_{\chi_g}}^{\mathfrak{S}-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-R_{\mathfrak{z}_{\chi_g}}^{\mathfrak{R}}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-R_{\mathfrak{z}_{\chi_g}}^{\mathfrak{S}}} - 1 \right)^{\omega_g} \right] \right), \end{array} \right) \quad (2)$$

Property 2. For any collection of CBCNFNs $\mathfrak{Z}_{\chi_g} = \left(\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Re+} + i\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Im+}, \mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Re-} + i\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Im-}, \mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Re+} + i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Im+}, \mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Re-} + i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Im-}, \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\Re+} + i\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\Im+}, \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\Re-} + i\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\Im-}, \mathcal{R}_{\mathfrak{Z}_{\chi_g}}^{\Re} + i\mathcal{R}_{\mathfrak{Z}_{\chi_g}}^{\Im} \right)$ $g = 1, 2, \dots, \mathfrak{L}$, then we obtain

(1) When $\mathfrak{z}_{\chi_g} = \mathfrak{z}_{\chi}$, $g = 1, 2, \dots, \mathfrak{L}$

$$CBCNFFPA^{\text{tcn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) = \mathfrak{z}_x$$

$$CBCNFFPA^{\text{tn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) = \mathfrak{z}_{\chi}$$

(2) When $\mathfrak{z}_{\chi_g} \leq \mathfrak{z}'_{\chi_g}$,

$$CBCNFFPA^{\text{tcn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_g) \leq CBCNFFPA^{\text{tcn}}(\mathfrak{B}'_1, \mathfrak{B}'_2 \dots \mathfrak{B}'_g)$$

$$CBCNFFPA^{\text{tn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_s) \leq CBCNFFPA^{\text{tn}}(\mathfrak{B}'_1, \mathfrak{B}'_2 \dots \mathfrak{B}'_s)$$

(3) When $\mathfrak{Z}_\chi^- = \min(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_\mathfrak{L})$ and $\mathfrak{Z}_\chi^+ = \max(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_\mathfrak{L})$

$$\mathfrak{z}_y^- \leq CBCNFFPA^{\text{tcn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) \leq \mathfrak{z}_y^+$$

$$\mathfrak{z}_y^- \leq CBCNFFPA^{\text{tn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) \leq \mathfrak{z}_y^+$$

Definition 16. For any collection of CBCNFNs $\mathfrak{Z}_{\chi_g} = (\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}+} + i\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}+}, \mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}-} + i\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}-}, \mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}+} + i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}+}, \mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}-} +$

$i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}-}, \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}+} + i\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}-} + i\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}-}, R_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}} + iR_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}}\big)g = 1, 2, \dots, \mathfrak{L}$, we define

$$CBCNFFPWA^{\text{tn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \oplus_{g=1}^{\text{tn}\mathfrak{L}} \frac{\mathfrak{J}(1 + \mathcal{N}(\mathfrak{Z}_g))}{\sum_{g=1}^{\mathfrak{L}} \mathfrak{J}(1 + \mathcal{N}(\mathfrak{Z}_g))} \mathfrak{Z}_g$$

$$CBCNFFPWA^{\text{tcn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \oplus_{g=1}^{\text{tcn}\mathfrak{L}} \frac{\mathfrak{J}(1 + \mathcal{N}(\mathfrak{Z}_g))}{\sum_{g=1}^{\mathfrak{L}} \mathfrak{J}(1 + \mathcal{N}(\mathfrak{Z}_g))} \mathfrak{Z}_g$$

Here, $CBCNFFPWA^{\text{tn}}$ denotes the CBCNFFPWA operator based on the $\mathcal{FTN}$, whereas $CBCNFFPWA^{\text{tcn}}$ denotes the CBCNFFPWA operator based on the $\mathcal{FTCN}$.

The following theorem provides the explicit expressions for the CBCNFFPWA operators. For convenience, let ϑ_g be the normalized power weight corresponding to $\mathfrak{Z}_g$, which is

$$\vartheta_g = \frac{\mathfrak{J}(1 + \mathcal{N}(\mathfrak{Z}_g))}{\sum_{g=1}^{\mathfrak{L}} \mathfrak{J}(1 + \mathcal{N}(\mathfrak{Z}_g))}, \quad g = 1, 2, \dots, \mathfrak{L}.$$

Clearly, $\vartheta_g \geq 0$, $\sum_{g=1}^{\mathfrak{L}} \vartheta_g = 1$. $\mathcal{N}(\mathfrak{Z}_g) = \sum_{i \neq g=1}^{\mathfrak{L}} S(\mathfrak{Z}_i, \mathfrak{Z}_g)$, where $S(\mathfrak{Z}_i, \mathfrak{Z}_g) = 1 - D(\mathfrak{Z}_i, \mathfrak{Z}_g)$.

Theorem 5. For any collection of CBCNFNs, the CBCNFFPWA operator produces a CBCNFN. That is $CBCNFFPWA^{\text{tcn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) =$

$$\left(\begin{array}{l} \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 - \mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}+}} - 1 \right)^{\vartheta_g} \right] \right) + i \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 - \mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}+}} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(-\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}-}} - 1 \right)^{\vartheta_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}-}} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}+}} - 1 \right)^{\vartheta_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}+}} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 + \mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}-}} - 1 \right)^{\vartheta_g} \right] \right) + i \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 + \mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}-}} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}+}} - 1 \right)^{\vartheta_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}+}} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 + \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}-}} - 1 \right)^{\vartheta_g} \right] \right) + i \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 + \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}-}} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 - R_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}}} - 1 \right)^{\vartheta_g} \right] \right) + i \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1 - R_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}}} - 1 \right)^{\vartheta_g} \right] \right), \end{array} \right)$$

$$CBCNFFPWA^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) =$$

$$\left(\begin{array}{l} \left(1 - \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{1 - \mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} - 1 \right)^{\vartheta_g} \right] \right) + i \left(1 - \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{1 - \mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(-\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{-\mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} - 1 \right)^{\vartheta_g} \right] \right) + i \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{-\mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{\mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} - 1 \right)^{\vartheta_g} \right] \right) + i \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{\mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(-1 + \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{1 + \mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} - 1 \right)^{\vartheta_g} \right] \right) + i \left(-1 + \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{1 + \mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{\mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} - 1 \right)^{\vartheta_g} \right] \right) + i \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{\mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(-1 + \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{1 + \mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} - 1 \right)^{\vartheta_g} \right] \right) + i \left(-1 + \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{1 + \mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-} - 1 \right)^{\vartheta_g} \right] \right), \\ \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{R_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}} - 1 \right)^{\vartheta_g} \right] \right) + i \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\alpha^{R_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}} - 1 \right)^{\vartheta_g} \right] \right), \end{array} \right)$$

Property 3. For any collection of CBCNFNs $\mathfrak{Z}_{\mathfrak{g}} = \left(\mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} + i\mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+}, \mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} + i\mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-}, \mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} + i\mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+}, \mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} + i\mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-}, \mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} + i\mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} + i\mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-}, R_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}} + iR_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}} \right) g = 1, 2, \dots, \mathfrak{L}$, we obtain

(1) When $\mathfrak{Z}_{\mathfrak{g}} = \mathfrak{Z}_{\chi}$, $g = 1, 2, \dots, \mathfrak{L}$

$$CBCNFFPWA^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \mathfrak{Z}_{\chi}$$

$$CBCNFFPWA^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \mathfrak{Z}_{\chi}$$

(2) When $\mathfrak{Z}_{\mathfrak{g}} \leq \mathfrak{Z}'_{\mathfrak{g}}$,

$$CBCNFFPWA^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) \leq CBCNFFPWA^{tn}(\mathfrak{Z}'_1, \mathfrak{Z}'_2 \dots \mathfrak{Z}'_{\mathfrak{L}})$$

$$CBCNFFPWA^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) \leq CBCNFFPWA^{tn}(\mathfrak{Z}'_1, \mathfrak{Z}'_2 \dots \mathfrak{Z}'_{\mathfrak{L}})$$

(3) When $\mathfrak{Z}_{\chi}^{-} = \min(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}})$ and $\mathfrak{Z}_{\chi}^{+} = \max(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}})$

$$\mathfrak{Z}_{\chi}^{-} \leq CBCNFFPWA^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) \leq \mathfrak{Z}_{\chi}^{+}$$

$$\mathfrak{Z}_{\chi}^{-} \leq CBCNFFPWA^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) \leq \mathfrak{Z}_{\chi}^{+}$$

Definition 17. For any collection of CBCNFNs $\mathfrak{Z}_{\mathfrak{g}} = \left(\mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} + i\mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+}, \mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} + i\mathcal{T}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-}, \mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} + i\mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+}, \mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} + i\mathcal{I}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-}, \mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}+} + i\mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}-} + i\mathcal{F}_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}-}, R_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{R}} + iR_{\mathfrak{Z}_{\mathfrak{g}}}^{\mathfrak{S}} \right) g = 1, 2, \dots, \mathfrak{L}$, we define

$$CBCNFFPG^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \bigotimes_{g=1}^{tn\mathfrak{L}} \frac{(1 + \mathcal{N}(\mathfrak{Z}_{\mathfrak{g}}))}{\sum_{g=1}^{\mathfrak{L}} (1 + \mathcal{N}(\mathfrak{Z}_{\mathfrak{g}}))} \mathfrak{Z}_{\mathfrak{g}}$$

$$CBCNFFPG^{tn}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \bigotimes_{g=1}^{tn\mathfrak{L}} \frac{(1 + \mathcal{N}(\mathfrak{Z}_{\mathfrak{g}}))}{\sum_{g=1}^{\mathfrak{L}} (1 + \mathcal{N}(\mathfrak{Z}_{\mathfrak{g}}))} \mathfrak{Z}_{\mathfrak{g}}$$

Here, $CBCNFFPWA^{tn}$ denotes the CBCNFFPWA operator based on the $\mathcal{FTN}$, whereas $CBCNFFPWA^{tn}$ denotes the CBCNFFPWA operator based on the $\mathcal{FTCN}$.

Theorem 6. For any collection of CBCNFNs, the CBCNFFPG operator produces a CBCNFN. That is

$$CBCNFFPG^{\text{tcn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) = \left(\begin{array}{l} \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{T}_{3\chi_g}^{\Re+} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{T}_{3\chi_g}^{\Im+} - 1 \right)^{\omega_g} \right] \right), \\ \left(-1 + \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1+\mathcal{T}_{3\chi_g}^{\Re-} - 1 \right)^{\omega_g} \right] \right) + i \left(-1 + \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1+\mathcal{T}_{3\chi_g}^{\Im-} - 1 \right)^{\omega_g} \right] \right), \\ \left(1 - \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-\mathcal{I}_{3\chi_g}^{\Re+} - 1 \right)^{\omega_g} \right] \right) + i \left(1 - \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-\mathcal{I}_{3\chi_g}^{\Im+} - 1 \right)^{\omega_g} \right] \right), \\ \left(-\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{I}_{3\chi_g}^{\Re-} - 1 \right)^{\omega_g} \right] \right) + i \left(-\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{I}_{3\chi_g}^{\Im-} - 1 \right)^{\omega_g} \right] \right), \\ \left(1 - \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-\mathcal{F}_{3\chi_g}^{\Re+} - 1 \right)^{\omega_g} \right] \right) + i \left(1 - \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-\mathcal{F}_{3\chi_g}^{\Im+} - 1 \right)^{\omega_g} \right] \right), \\ \left(-\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{F}_{3\chi_g}^{\Re-} - 1 \right)^{\omega_g} \right] \right) + i \left(-\lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{F}_{3\chi_g}^{\Im-} - 1 \right)^{\omega_g} \right] \right), \\ \left(1 - \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-R_{3\chi_g}^{\Re} - 1 \right)^{\omega_g} \right] \right) + i \left(1 - \lg_{\alpha} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-R_{3\chi_g}^{\Im} - 1 \right)^{\omega_g} \right] \right), \end{array} \right) \quad (3)$$

$$CBCNFFPG^{\text{tn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) = \left(\begin{array}{l} \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{T}_{3\chi_g}^{\Re+}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{\mathcal{T}_{3\chi_g}^{\Im+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1+\mathcal{T}_{3\chi_g}^{\Re-}} - 1 \right)^{\omega_g} \right] \right) + i \left(-1 + \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1+\mathcal{T}_{3\chi_g}^{\Im-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-\mathcal{I}_{3\chi_g}^{\Re+}} - 1 \right)^{\omega_g} \right] \right) + i \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-\mathcal{I}_{3\chi_g}^{\Im+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{I}_{3\chi_g}^{\Re-}} - 1 \right)^{\omega_g} \right] \right) + i \left(-\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{I}_{3\chi_g}^{\Im-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-\mathcal{F}_{3\chi_g}^{\Re+}} - 1 \right)^{\omega_g} \right] \right) + i \left(1 - \lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{1-\mathcal{F}_{3\chi_g}^{\Im+}} - 1 \right)^{\omega_g} \right] \right), \\ \left(-\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{F}_{3\chi_g}^{\Re-}} - 1 \right)^{\omega_g} \right] \right) + i \left(-\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-\mathcal{F}_{3\chi_g}^{\Im-}} - 1 \right)^{\omega_g} \right] \right), \\ \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-R_{3\chi_g}^{\Re}} - 1 \right)^{\omega_g} \right] \right) + i \left(\lg_{\mathfrak{a}} \left[1 + \prod_{g=1}^{\mathfrak{L}} \left(\mathfrak{a}^{-R_{3\chi_g}^{\Im}} - 1 \right)^{\omega_g} \right] \right), \end{array} \right) \quad (4)$$

Property 4. For any collection of CBCNFNs $\mathfrak{Z}_{\chi_{\mathfrak{g}}} = \left(\mathcal{T}_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Re+} + i\mathcal{T}_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Im+}, \mathcal{T}_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Re-} + i\mathcal{T}_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Im-}, \mathcal{F}_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Re+} + i\mathcal{F}_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Im+}, \mathcal{F}_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Re-} + i\mathcal{F}_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Im-}, R_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Re} + iR_{\mathfrak{Z}_{\chi_{\mathfrak{g}}}}^{\Im} \right)$ $\mathfrak{g} = 1, 2, \dots, \mathfrak{L}$, we obtain

(1) When $\mathfrak{z}_{\chi_g} = \mathfrak{z}_{\chi}$, $g = 1, 2, \dots, \mathfrak{L}$

$$CBCNFFPG^{\text{tcn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) = \mathfrak{z}_{\chi}$$

$$CBCNFFPG^{\text{tn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{L}}) = \mathfrak{z}_{\mathfrak{y}}$$

(2) When $\mathfrak{z}_{\chi_g} \leq \mathfrak{z}'_{\chi_g}$,

$$CBCNFFPG^{\text{tcn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{z}}) \leq CBCNFFPG^{\text{tcn}}(\mathfrak{z}'_1, \mathfrak{z}'_2 \dots \mathfrak{z}'_{\mathfrak{z}})$$

$$CBCNFFPG^{\text{tn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_{\mathfrak{s}}) \leq CBCNFFPG^{\text{tn}}(\mathfrak{z}'_1, \mathfrak{z}'_2 \dots \mathfrak{z}'_{\mathfrak{s}})$$

(3) When $\mathfrak{Z}_y^- = \min(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}})$ and $\mathfrak{Z}_y^+ = \max(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}})$

$$\mathfrak{z}_\chi^- \leq CBCNFFPG^{\text{tcn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_\mathfrak{L}) \leq \mathfrak{z}_\chi^+$$

$$\mathfrak{z}_\chi^- \leq C B C N F F P G^{\text{tn}}(\mathfrak{z}_1, \mathfrak{z}_2 \dots \mathfrak{z}_\mathfrak{L}) \leq \mathfrak{z}_\chi^+$$

Definition 18. For any collection of CBCNFNs $\mathfrak{Z}_{\chi_g} = \left(\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Re+} + i\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Im+}, \mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Re-} + i\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Im-}, \mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Re+} + i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Im+}, \mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Re-} + i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Im-}, \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\Re+} + i\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\Im+}, \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\Re-} + i\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\Im-}, R_{\mathfrak{Z}_{\chi_g}}^{\Re} + iR_{\mathfrak{Z}_{\chi_g}}^{\Im} \right)$ $g = 1, 2, \dots, \mathfrak{L}$, we define

Here, $CBCNFFPWA^{\text{tn}}$ denotes the CBCNFFPWA operator based on the $\mathcal{FTN}$, whereas $CBCNFFPWA^{\text{tcn}}$ denotes the CBCNFFPWA operator based on the $\mathcal{FTCN}$.

$$\left(\begin{aligned} & \left(\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{\mathcal{T}_{3\chi_g}^{\Re+} - 1 \right)^{\vartheta_g} \right] \right) + i \left(\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{\mathcal{T}_{3\chi_g}^{\Im+} - 1 \right)^{\vartheta_g} \right] \right), \\ & \left(-1 + \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1+\mathcal{T}_{3\chi_g}^{\Re-} - 1 \right)^{\vartheta_g} \right] \right) + i \left(-1 + \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1+\mathcal{T}_{3\chi_g}^{\Im-} - 1 \right)^{\vartheta_g} \right] \right), \\ & \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-\mathcal{T}_{3\chi_g}^{\Re+} - 1 \right)^{\vartheta_g} \right] \right) + i \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-\mathcal{T}_{3\chi_g}^{\Im+} - 1 \right)^{\vartheta_g} \right] \right), \\ & \left(-\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{-\mathcal{T}_{3\chi_g}^{\Re-} - 1 \right)^{\vartheta_g} \right] \right) + i \left(-\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{-\mathcal{T}_{3\chi_g}^{\Im-} - 1 \right)^{\vartheta_g} \right] \right), \\ & \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-\mathcal{F}_{3\chi_g}^{\Re+} - 1 \right)^{\vartheta_g} \right] \right) + i \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-\mathcal{F}_{3\chi_g}^{\Im+} - 1 \right)^{\vartheta_g} \right] \right), \\ & \left(-\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{-\mathcal{F}_{3\chi_g}^{\Re-} - 1 \right)^{\vartheta_g} \right] \right) + i \left(-\lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{-\mathcal{F}_{3\chi_g}^{\Im-} - 1 \right)^{\vartheta_g} \right] \right), \\ & \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-R_{3\chi_g}^{\Re} - 1 \right)^{\vartheta_g} \right] \right) + i \left(1 - \lg_a \left[1 + \prod_{g=1}^{\mathcal{L}} \left(a^{1-R_{3\chi_g}^{\Im} - 1 \right)^{\vartheta_g} \right] \right), \end{aligned} \right)$$

Property 5. For any collection of CBCNFNs $\mathfrak{Z}_{\chi_g} = (\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Re+} + i\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Im+}, \mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Re-} + i\mathcal{T}_{\mathfrak{Z}_{\chi_g}}^{\Im-}, \mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Re+} + i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Im+}, \mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Re-} + i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\Im-})$

$i\mathcal{I}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}-}, \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}+} + i\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}-} + i\mathcal{F}_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}-}, R_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{R}} + iR_{\mathfrak{Z}_{\chi_g}}^{\mathfrak{S}} \Big) g = 1, 2, \dots, \mathfrak{L}$, we obtain

(1) When $\mathfrak{Z}_{\chi_g} = \mathfrak{Z}_{\chi}$, $g = 1, 2, \dots, \mathfrak{L}$

$$CBCNFFPWG^{\text{tcn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \mathfrak{Z}_{\chi}$$

$$CBCNFFPWG^{\text{tn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) = \mathfrak{Z}_{\chi}$$

(2) When $\mathfrak{Z}_{\chi_g} \leq \mathfrak{Z}'_{\chi_g}$,

$$CBCNFFPWG^{\text{tcn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) \leq CBCNFFPWG^{\text{tcn}}(\mathfrak{B}'_1, \mathfrak{B}'_2 \dots \mathfrak{B}'_{\mathfrak{L}})$$

$$CBCNFFPWG^{\text{tn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) \leq CBCNFFPWG^{\text{tn}}(\mathfrak{B}'_1, \mathfrak{B}'_2 \dots \mathfrak{B}'_{\mathfrak{L}})$$

(3) When $\mathfrak{Z}_{\chi}^{-} = \min(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}})$ and $\mathfrak{Z}_{\chi}^{+} = \max(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}})$

$$\mathfrak{Z}_{\chi}^{-} \leq CBCNFFPWG^{\text{tcn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) \leq \mathfrak{Z}_{\chi}^{+}$$

$$\mathfrak{Z}_{\chi}^{-} \leq CBCNFFPWG^{\text{tn}}(\mathfrak{Z}_1, \mathfrak{Z}_2 \dots \mathfrak{Z}_{\mathfrak{L}}) \leq \mathfrak{Z}_{\chi}^{+}$$

5. MADM algorithm for Circular Bipolar Complex Neutrosophic Fuzzy Sets

Step 1: Construct the decision matrix by arranging the CBCNFN values as follows: $D_{\eta} = \left(\mathcal{T}_{\mathfrak{D}_{\eta}}^{\mathfrak{R}+} + i\mathcal{T}_{\mathfrak{D}_{\eta}}^{\mathfrak{S}+}, \mathcal{T}_{\mathfrak{D}_{\eta}}^{\mathfrak{R}-} + i\mathcal{T}_{\mathfrak{D}_{\eta}}^{\mathfrak{S}-}, \mathcal{I}_{\mathfrak{D}_{\eta}}^{\mathfrak{R}+} + i\mathcal{I}_{\mathfrak{D}_{\eta}}^{\mathfrak{S}+}, \mathcal{I}_{\mathfrak{D}_{\eta}}^{\mathfrak{R}-} + i\mathcal{I}_{\mathfrak{D}_{\eta}}^{\mathfrak{S}-}, \mathcal{F}_{\mathfrak{D}_{\eta}}^{\mathfrak{R}+} + i\mathcal{F}_{\mathfrak{D}_{\eta}}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{D}_{\eta}}^{\mathfrak{R}-} + i\mathcal{F}_{\mathfrak{D}_{\eta}}^{\mathfrak{S}-}, R_{\mathfrak{D}_{\eta}}^{\mathfrak{R}} + iR_{\mathfrak{D}_{\eta}}^{\mathfrak{S}} \right)$, denotes the CBCNFNs.

Step 2: After performing the required normalization, obtain the weighted decision matrix

$$bD' = \left[\begin{array}{ll} \left(\mathcal{T}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}+} + i\mathcal{T}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}+}, \mathcal{T}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}-} + i\mathcal{T}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}-}, \mathcal{I}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}+} + i\mathcal{I}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}+}, \mathcal{I}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}-} + i\mathcal{I}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}-}, \right. & \text{benefit type} \\ \left. \mathcal{F}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}+} + i\mathcal{F}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}-} + i\mathcal{F}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}-}, R_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}} + iR_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}} \right) & \\ \mathcal{N} \left(\mathcal{T}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}+} + i\mathcal{T}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}+}, \mathcal{T}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}-} + i\mathcal{T}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}-}, \mathcal{I}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}+} + i\mathcal{I}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}+}, \mathcal{I}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}-} + i\mathcal{I}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}-}, \right. & \text{cost type} \\ \left. \mathcal{F}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}+} + i\mathcal{F}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}+}, \mathcal{F}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}-} + i\mathcal{F}_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}-}, R_{\mathfrak{D}_{\eta_i}}^{\mathfrak{R}} + iR_{\mathfrak{D}_{\eta_i}}^{\mathfrak{S}} \right) & \end{array} \right]$$

Normalization is not required for benefit-type criteria and for cost-type criteria the corresponding normalization operation is applied.

Step 3: Following the required normalization, obtain the weighted matrix using the proposed CBCNFFPA and CBCNFFPG aggregation operators given in Equations (1) and (3).

Step 4: Compute the score function for each aggregated CBCNFN.

Step 5: Rank the alternatives in descending order according to their score values S_i . The alternative with the highest score is ranked first. The step-by-step procedure of the proposed decision-making approach is illustrated in Figure 1.

5.1 EV Charging Station Selection Using CBCNFS Frank Power Aggregation Operators

The proposed CBCNFS framework and Frank Power aggregation operators are demonstrated through a hypothetical MADM problem involving an electric vehicle charging station. The purpose of this numerical illustration is to demonstrate how the proposed framework represents and aggregates multidimensional uncertain information under the considered decision criteria. Modern transportation systems involve dynamic and uncertain decision environments due to the rapid growth of electric mobility. Because of varying criteria such as availability, waiting time, charging efficiency, and cost, EV users may face difficulties when selecting appropriate charging stations. Incomplete information, fluctuating traffic conditions, and cyclical demand patterns are other sources of uncertainty.

To illustrate the applicability of the proposed method, five representative hypothetical EV charging-station alternatives are considered. Ξ_1 : Quick urban charging station, Ξ_2 : Highway fast charging station, Ξ_3 : Commercial area charging station, Ξ_4 : Neighborhood residential charging unit, and Ξ_5 : Solar-powered environmentally friendly charging station. These alternatives are typical EV charging facilities found in contemporary smart cities.

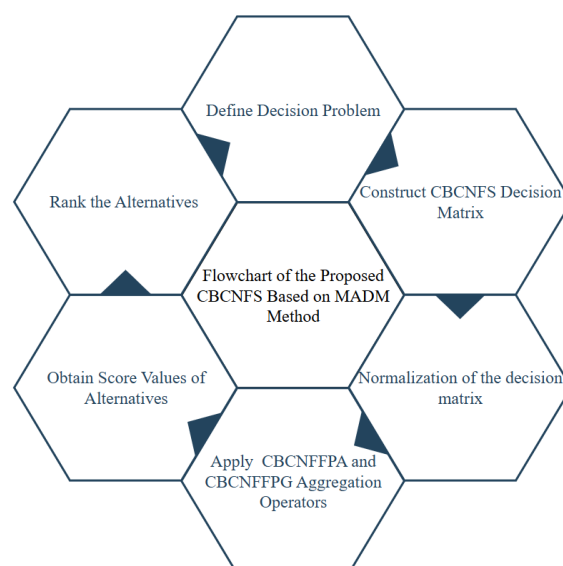


Fig. 1. Flowchart of the proposed CBCNFS-based MADM algorithm

As the main aim of this study is to exhibit the utility and behaviour of the introduced CBCNFS aggregation technique, the decision matrix considered in this example is illustrative and was not obtained from observations of actual charging stations. Hence, in this example, five alternatives have been considered as five representative charging station profiles that correspond to the urban, highway, commercial area, residential, and solar-powered charging stations, respectively. The evaluation values of Table 1 have been devised to express differences between the alternatives based on the selected criteria such as distance, charging cost, reliability, charging time, and waiting time. The illustrative values were constructed to reflect plausible relative performance differences among the representative charging station profiles with respect to the selected criteria, rather than measurements collected from actual charging stations. These values have deliberately been allocated such that the proposed structure of CBCNFS can express both positive and negative information, indeterminacy, complex-valued information, and circular uncertainty simultaneously. The following five criteria are used to assess each charging station:

- Distance ($\mathbb{C}_1$): It shows how close the user is to the charging station. Convenience and energy efficiency dictate a shorter distance.
- Expense ($\mathbb{C}_2$): It refers to the financial cost of charging. Users prefer lower costs.
- Availability and Dependability ($\mathbb{C}_3$): It assesses the station's operational consistency, including fault frequency and uptime.
- Speed of Charging ($\mathbb{C}_4$): It determines how quickly the car may be charged at the station. Usability is enhanced by faster charging.
- Waiting Duration ($\mathbb{C}_5$): It shows the anticipated length of the station's line. Better service availability is reflected in shorter wait times.

Since the numerical illustration is intended to demonstrate the proposed decision-making process, the criterion weights are specified hypothetically to represent their relative importance. These weights are not obtained from an actual survey of experts.

The designed decision matrix of the CBCNF shown in Table 1 is synthetically constructed for illustrative purposes, and no expert judgments, survey responses, field data, or other real-world datasets were used. Therefore, the linguistic scale and the linguistic term to CBCNF transformation process will not be applied in this illustration. CBCNF values are determined directly using the intervals of possible values of the CBCNFS model proposed. The positive and negative components represent favorable and unfavorable evaluations, respectively. The real and imaginary components represent two-dimensional complex information, while the T, I, and F components denote truth, indeterminacy, and falsity, respectively. The radius component represents circular uncertainty. The values have been

taken so that all five alternatives show different levels of performance across the five criteria. Thus, the values in Table 1 should be interpreted as constructed CBCNF test values for demonstrating the mathematical behaviour and ranking procedure of the proposed operators, rather than as empirical observations or expert-based evaluations.

The entries in Table 1 are synthetically constructed CBCNF values for demonstrating the proposed decision-making process. Positive and negative complex components correspond to the assessments, whereas the final component represents the radius of the circle. In this regard, the normalized weight vector $(0.25, 0.20, 0.15, 0.15, 0.25)$ has been used to construct an illustrative scenario representing the relative importance of the criteria, with greater weights assigned to distance and waiting time because both affect the user's ease of access. Cost of charging is also considered an important criterion, whereas availability, reliability, and charging speed are assigned moderate weights. It is not argued that the aforementioned criteria have the same weight in reality but are used to construct an illustrative scenario.

First, using the provided CBCNFS data, the decision matrix is initially constructed, as shown in Table 1. The criteria values for each alternative are then combined using the PA aggregation operator described in Equation (1) and (3). The computations are performed by considering the interactions among the criteria for $\alpha = 2$. So, Table 2 presents the aggregated results.

The score values of each choice are calculated using the scoring function specified in Definition 10 after the aggregated values have been obtained. The alternatives can be ranked and compared using these score values. Table 3 displays the calculated score values.

Table 1

Decision matrix of CBCNF evaluations

	C_1	C_2	C_3
Ξ_1	$\begin{pmatrix} 0.51 + i0.31 & -0.71 - i0.21 \\ 0.31 + i0.71 & -0.51 - i0.81 \\ 0.21 + i0.91 & -0.41 - i0.41 \\ 0.81 + i0.51 \end{pmatrix}$	$\begin{pmatrix} 0.52 + i0.32 & -0.72 - i0.22 \\ 0.32 + i0.72 & -0.52 - i0.82 \\ 0.22 + i0.92 & -0.42 - i0.42 \\ 0.82 + i0.52 \end{pmatrix}$	$\begin{pmatrix} 0.53 + i0.33 & -0.73 - i0.23 \\ 0.33 + i0.73 & -0.53 - i0.83 \\ 0.23 + i0.93 & -0.43 - i0.43 \\ 0.83 + i0.53 \end{pmatrix}$
Ξ_2	$\begin{pmatrix} 0.21 + i0.41 & -0.31 - i0.51 \\ 0.61 + i0.31 & -0.51 - i0.71 \\ 0.31 + i0.21 & -0.81 - i0.61 \\ 0.51 + i0.21 \end{pmatrix}$	$\begin{pmatrix} 0.22 + i0.42 & -0.32 - i0.52 \\ 0.62 + i0.32 & -0.52 - i0.72 \\ 0.32 + i0.22 & -0.82 - i0.62 \\ 0.52 + i0.22 \end{pmatrix}$	$\begin{pmatrix} 0.23 + i0.43 & -0.33 - i0.53 \\ 0.63 + i0.33 & -0.53 - i0.73 \\ 0.33 + i0.23 & -0.83 - i0.63 \\ 0.53 + i0.23 \end{pmatrix}$
Ξ_3	$\begin{pmatrix} 0.51 + i0.61 & -0.61 - i0.71 \\ 0.71 + i0.81 & -0.51 - i0.41 \\ 0.81 + i0.31 & -0.91 - i0.81 \\ 0.51 + i0.21 \end{pmatrix}$	$\begin{pmatrix} 0.52 + i0.62 & -0.62 - i0.72 \\ 0.72 + i0.82 & -0.52 - i0.42 \\ 0.82 + i0.32 & -0.92 - i0.82 \\ 0.52 + i0.22 \end{pmatrix}$	$\begin{pmatrix} 0.53 + i0.63 & -0.63 - i0.73 \\ 0.73 + i0.83 & -0.53 - i0.43 \\ 0.83 + i0.33 & -0.93 - i0.83 \\ 0.53 + i0.23 \end{pmatrix}$
Ξ_4	$\begin{pmatrix} 0.51 + i0.61 & -0.61 - i0.81 \\ 0.81 + i0.71 & -0.51 - i0.21 \\ 0.81 + i0.51 & -0.21 - i0.31 \\ 0.81 + i0.91 \end{pmatrix}$	$\begin{pmatrix} 0.52 + i0.62 & -0.62 - i0.82 \\ 0.82 + i0.72 & -0.52 - i0.22 \\ 0.82 + i0.52 & -0.22 - i0.32 \\ 0.82 + i0.92 \end{pmatrix}$	$\begin{pmatrix} 0.53 + i0.63 & -0.63 - i0.83 \\ 0.83 + i0.73 & -0.53 - i0.23 \\ 0.83 + i0.53 & -0.23 - i0.33 \\ 0.83 + i0.93 \end{pmatrix}$
Ξ_5	$\begin{pmatrix} 0.71 + i0.51 & -0.61 - i0.41 \\ 0.51 + i0.31 & -0.81 - i0.21 \\ 0.61 + i0.51 & -0.21 - i0.71 \\ 0.61 + i0.21 \end{pmatrix}$	$\begin{pmatrix} 0.72 + i0.52 & -0.62 - i0.42 \\ 0.52 + i0.32 & -0.82 - i0.22 \\ 0.62 + i0.52 & -0.22 - i0.72 \\ 0.62 + i0.22 \end{pmatrix}$	$\begin{pmatrix} 0.73 + i0.53 & -0.63 - i0.43 \\ 0.53 + i0.33 & -0.83 - i0.23 \\ 0.63 + i0.53 & -0.23 - i0.73 \\ 0.63 + i0.23 \end{pmatrix}$
	C_4	C_5	
Ξ_1	$\begin{pmatrix} 0.54 + i0.34 & -0.74 - i0.24 \\ 0.34 + i0.74 & -0.54 - i0.84 \\ 0.24 + i0.94 & -0.44 - i0.44 \\ 0.84 + i0.54 \end{pmatrix}$	$\begin{pmatrix} 0.55 + i0.35 & -0.75 - i0.25 \\ 0.35 + i0.75 & -0.55 - i0.85 \\ 0.25 + i0.95 & -0.45 - i0.45 \\ 0.85 + i0.55 \end{pmatrix}$	

Table 1
Continued

	$\mathbb{C}_4$	$\mathbb{C}_5$
Ξ_2	$\begin{pmatrix} 0.24 + i0.44 & -0.34 - i0.54 \\ 0.64 + i0.34 & -0.54 - i0.74 \\ 0.34 + i0.24 & -0.84 - i0.64 \\ 0.54 + i0.24 \end{pmatrix}$	$\begin{pmatrix} 0.25 + i0.45 & -0.35 - i0.55 \\ 0.65 + i0.35 & -0.55 - i0.75 \\ 0.35 + i0.25 & -0.85 - i0.65 \\ 0.55 + i0.25 \end{pmatrix}$
Ξ_3	$\begin{pmatrix} 0.54 + i0.64 & -0.64 - i0.74 \\ 0.74 + i0.84 & -0.54 - i0.44 \\ 0.84 + i0.34 & -0.94 - i0.84 \\ 0.54 + i0.24 \end{pmatrix}$	$\begin{pmatrix} 0.55 + i0.65 & -0.65 - i0.75 \\ 0.75 + i0.85 & -0.55 - i0.45 \\ 0.85 + i0.35 & -0.95 - i0.85 \\ 0.55 + i0.25 \end{pmatrix}$
Ξ_4	$\begin{pmatrix} 0.54 + i0.64 & -0.64 - i0.84 \\ 0.84 + i0.74 & -0.54 - i0.24 \\ 0.84 + i0.54 & -0.24 - i0.34 \\ 0.84 + i0.94 \end{pmatrix}$	$\begin{pmatrix} 0.55 + i0.65 & -0.65 - i0.85 \\ 0.85 + i0.75 & -0.55 - i0.25 \\ 0.85 + i0.55 & -0.25 - i0.35 \\ 0.85 + i0.95 \end{pmatrix}$
Ξ_5	$\begin{pmatrix} 0.74 + i0.54 & -0.64 - i0.44 \\ 0.54 + i0.34 & -0.84 - i0.24 \\ 0.64 + i0.54 & -0.24 - i0.74 \\ 0.64 + i0.24 \end{pmatrix}$	$\begin{pmatrix} 0.75 + i0.55 & -0.65 - i0.45 \\ 0.55 + i0.35 & -0.85 - i0.25 \\ 0.65 + i0.55 & -0.25 - i0.75 \\ 0.65 + i0.25 \end{pmatrix}$

Table 2
Aggregated matrix

	CBCNFFPA operator
Ξ_1	$\begin{pmatrix} 0.530180 + i0.330117 & -0.729895 + i(-0.229599) \\ 0.329731 + i0.729895 & -0.530180 + i(-0.830555) \\ 0.229599 + i0.929924 & -0.430143 + i(-0.430143) \\ 0.830555 + i0.530180 \end{pmatrix}$
Ξ_2	$\begin{pmatrix} 0.230098 + i0.430143 & -0.329731 + i(-0.529844) \\ 0.629874 + i0.329731 & -0.530180 + i(-0.730336) \\ 0.329731 + i0.229599 & -0.830555 + i(-0.630237) \\ 0.530180 + i0.230098 \end{pmatrix}$
Ξ_3	$\begin{pmatrix} 0.530180 + i0.630237 & -0.629874 + i(-0.729895) \\ 0.729895 + i0.829911 & -0.530180 + i(-0.430143) \\ 0.829911 + i0.329731 & -0.931427 + i(-0.830555) \\ 0.530180 + i0.230098 \end{pmatrix}$
Ξ_4	$\begin{pmatrix} 0.530180 + i0.630237 & -0.629874 + i(-0.829911) \\ 0.829911 + i0.729895 & -0.530180 + i(-0.230098) \\ 0.829911 + i0.529844 & -0.230098 + i(-0.330117) \\ 0.830555 + i0.931427 \end{pmatrix}$
Ξ_5	$\begin{pmatrix} 0.730336 + i0.530180 & -0.629874 + i(-0.429801) \\ 0.529844 + i0.329731 & -0.830555 + i(-0.230098) \\ 0.629874 + i0.529844 & -0.230098 + i(-0.730336) \\ 0.630237 + i0.230098 \end{pmatrix}$

CBCNFFPG operator	
Ξ_1	$\begin{pmatrix} 0.529844 + i0.329731 & -0.730336 + i(-0.230098) \\ 0.330117 + i0.730336 & -0.529844 + i(-0.829911) \\ 0.230098 + i0.931427 & -0.429801 + i(-0.429801) \\ 0.830555 + i0.530180 & \end{pmatrix}$
Ξ_2	$\begin{pmatrix} 0.229599 + i0.429801 & -0.330117 + i(-0.530180) \\ 0.630237 + i0.330117 & -0.529844 + i(-0.729895) \\ 0.330117 + i0.230098 & -0.829911 + i(-0.629874) \\ 0.530180 + i0.230098 & \end{pmatrix}$
Ξ_3	$\begin{pmatrix} 0.529844 + i0.629874 & -0.630237 + i(-0.730336) \\ 0.730336 + i0.830555 & -0.529844 + i(-0.429801) \\ 0.830555 + i0.330117 & -0.929924 + i(-0.829911) \\ 0.530180 + i0.230098 & \end{pmatrix}$
Ξ_4	$\begin{pmatrix} 0.529844 + i0.629874 & -0.630237 + i(-0.830555) \\ 0.830555 + i0.730336 & -0.529844 + i(-0.229599) \\ 0.830555 + i0.530180 & -0.229599 + i(-0.329731) \\ 0.830555 + i0.931427 & \end{pmatrix}$
Ξ_5	$\begin{pmatrix} 0.729895 + i0.529844 & -0.630237 + i(-0.430143) \\ 0.530180 + i0.330117 & -0.829911 + i(-0.229599) \\ 0.630237 + i0.530180 & -0.229599 + i(-0.729895) \\ 0.630237 + i0.230098 & \end{pmatrix}$

Table 3
Score values of the aggregated results

	CBCNFFPA operator	CBCNFFPG operator
Ξ_1	-0.421436	-0.421638
Ξ_2	-0.371418	-0.371519
Ξ_3	-0.457242	-0.457298
Ξ_4	-0.450100	-0.450247
Ξ_5	-0.335705	-0.335764

Table 4
Ranking of alternatives using CBCNFFPA and CBCNFFPG operators

	Ranking	Best alternative
CBCNFFPA operator	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	Ξ_5
CBCNFFPG operator	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	Ξ_5

The alternatives are arranged in descending order according to their computed score values. Table 4 displays the ranking of each alternative.

The alternatives are assessed and ranked utilizing both the PA and PG aggregation operators using the proposed MADM technique. The proposed score function is used to compute the aggregated data and obtain the corresponding score values. The results presented in the tables show that the alternative Ξ_5 consistently receives the highest score, making it the highest-ranked alternative in the considered numerical illustration.

Figure 2 shows the comparison of score values for each alternative using the CBCNFFPA and CBCNFFPG operators.

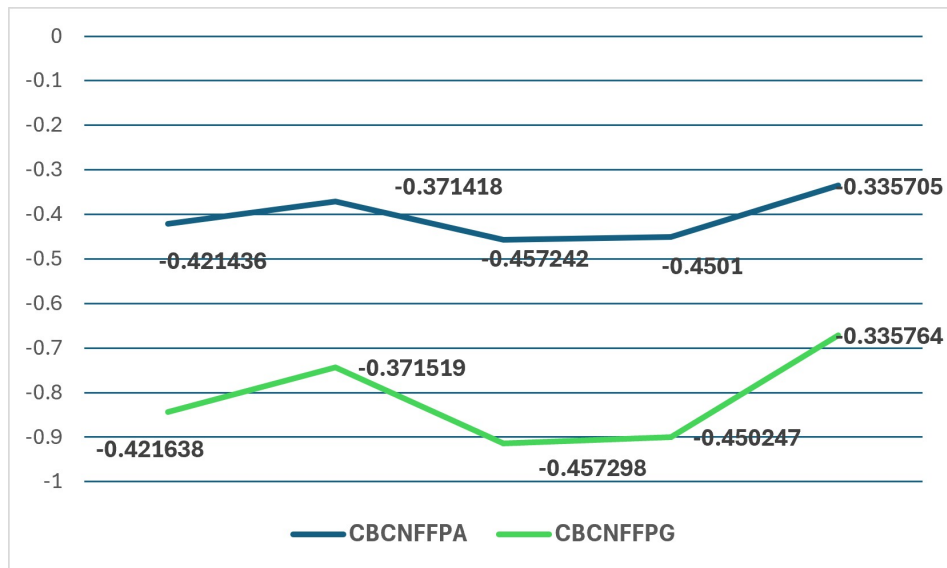


Fig. 2. Graphical Representation of Score Values of Alternatives Using PA and PG Operators

Additionally, a sensitivity analysis is performed by varying the parameter $\alpha = 2, 3, 4, 5, 6, 7, 8, 9, 10$. The results, as displayed in the associated tables, show that the ranking order of the alternatives stays constant for all values of α , despite minor differences in the numerical score values.

5.2 Sensitivity Analysis

A sensitivity analysis was performed by varying the parameter α from 2 to 10 to evaluate the stability of the proposed decision-making model. The parameter α , which plays an important role in the FAO, affects the interaction behaviour between criteria. The results from both CBCNFFPA and CBCNFFPG operators show that while there are minor numerical changes in the score values as α increases, the overall ranking of the alternatives stays the same. As shown in Table 5, alternative Ξ_5 consistently receives the highest score. The results demonstrate ranking stability with respect to variations in the Frank parameter α . Therefore, for the investigated values of the Frank parameter α in the present numerical example, the proposed framework demonstrates ranking stability and provides consistent preference orderings. The rankings obtained for the considered numerical illustration are

Table 5

Sensitivity analysis of CBCNFFPA and CBCNFFPG operators with respect to the parameter α

Parameter	CBCNFFPA operator	CBCNFFPG operator
$\alpha = 2$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$
$\alpha = 3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$
$\alpha = 4$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$
$\alpha = 5$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$
$\alpha = 6$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$
$\alpha = 7$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$
$\alpha = 8$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$
$\alpha = 9$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$
$\alpha = 10$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$	$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$

$\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$ for the CBCNFFPA operator and $\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$ for the CBCNFFPG operator. These rankings provide the ordering of the alternatives under the proposed CBCNF-based decision-making framework for the considered numerical illustration. The alternative

Ξ_5 emerges as the most preferred, while Ξ_3 is the least preferred alternative according to the proposed decision-making environment. Therefore, within the considered numerical illustration, Ξ_5 obtains the highest score under the proposed framework. In other words, the proposed approach helps aggregate the uncertainties and complexities of the decision-making problem by aggregating the information in one decision value rather than judging individual criteria separately. This is because the preference for Ξ_5 does not mean that it must be the most preferred alternative in all individual criteria. Rather, this preference arises based on its cumulative performance over all the criteria. Therefore, this allows the decision maker to derive an overall preference based on the multiple dimensions of the information available. There are various implications of the proposed framework on EV charging stations planning. First, the proposed method can be used to compare different charging-station alternatives under uncertain information. Second, the use of the proposed framework allows decision-makers to consider different types of information within the same decision-making framework instead of using one representation of information. In practical applications, the resulting ranking could potentially support resource-allocation and infrastructure-planning decisions after validation with real-world data.

The findings indicate that the ordering of the alternatives remains unchanged for all considered values of the Frank parameter α . This indicates that, for the particular illustrative decision problem, the variations in the parameter influence the process of aggregation but not sufficiently to change the ordering of the alternatives. In other words, for each aggregation operator, the relative ordering of the alternatives remains unchanged across the considered Frank parameter values, although the CBCNFFPA and CBCNFFPG operators may yield different ranking sequences. The CBCNFFPA and CBCNFFPG operators may, however, produce different rankings from each other because of differences in their aggregation behaviour. Indeed, the parameter can influence the values of the aggregated evaluations and their score values, although not so much that it would lead to any changes in the order of the alternatives. Thus, the identical ranking implies limited practical effect of the parameter on the ranking in the current numerical example and the investigated range of the parameter values.

5.2.1 Sensitivity with Respect to Criterion Weights

In addition to examining the effect of the Frank parameter, the stability of the proposed decision-making model was further investigated with respect to variations in the criterion weights. The original weight vector was considered as $\mathbf{w}^{(0)} = (0.25, 0.20, 0.15, 0.15, 0.25)$. To analyse the effect of changes in the importance assigned to individual criteria, several positive and negative perturbations were introduced. For a perturbation vector $\delta = (\delta_1, \delta_2, \dots, \delta_5)$, the modified criterion weights were obtained as

$$w'_j = \frac{w_j + \delta_j}{\sum_{k=1}^5 (w_k + \delta_k)}, \quad j = 1, 2, \dots, 5,$$

provided that $w_j + \delta_j > 0$ for all criteria. This normalization ensures that all modified weights remain positive and satisfy $\sum_{j=1}^5 w'_j = 1$. To investigate the influence of changes in criterion importance on the proposed decision-making results, several weight perturbation scenarios are considered. Let the original weight vector be denoted by

$$\omega = (\omega_1, \omega_2, \dots, \omega_n), \quad \sum_{j=1}^n \omega_j = 1.$$

For each perturbation scenario, the weights are modified by introducing a perturbation vector

$$\delta = (\delta_1, \delta_2, \dots, \delta_n).$$

The perturbed weights are subsequently normalized according to

$$\omega'_j = \frac{\omega_j + \delta_j}{\sum_{k=1}^n (\omega_k + \delta_k)},$$

where $\delta_j > 0$ represents an increase in the importance of the j th criterion, while $\delta_j < 0$ represents a decrease.

The original weight vector is given by

$$\omega^{(0)} = (0.25, 0.20, 0.15, 0.15, 0.25).$$

The following perturbation cases are evaluated.

- Original weights: The original weight vector remains unchanged:

$$\omega^{(0)} = (0.25, 0.20, 0.15, 0.15, 0.25).$$

- Increase in $\mathbb{C}_1$: A perturbation is introduced into the first criterion: $\delta = (0.05, 0, 0, 0, 0)$. After normalization, the resultant weight vector will be approximately

$$\omega^{(1)} = (0.286, 0.190, 0.143, 0.143, 0.238).$$

- Decrease in $\mathbb{C}_1$: A negative perturbation is applied to the first criterion: $\delta = (-0.05, 0, 0, 0, 0)$. After normalization, the resultant weight vector will be approximately

$$\omega^{(2)} = (0.211, 0.211, 0.158, 0.158, 0.263).$$

- Increase in $\mathbb{C}_5$: A positive perturbation is applied to the fifth criterion: $\delta = (0, 0, 0, 0, 0.05)$. After normalization, the resultant weight vector will be approximately

$$\omega^{(3)} = (0.238, 0.190, 0.143, 0.143, 0.286).$$

- Increase in the middle criteria: Positive perturbations are applied to the middle criteria: $\delta = (0, 0.03, 0.03, 0.03, 0)$. After normalization, the resultant weight vector will be approximately

$$\omega^{(4)} = (0.229, 0.211, 0.165, 0.165, 0.229).$$

- Mixed perturbation: A combination of positive and negative perturbations is considered: $\delta = (0.05, -0.03, 0.02, -0.02, -0.02)$. After normalization, the resultant weight vector will be approximately

$$\omega^{(5)} = (0.300, 0.170, 0.170, 0.130, 0.230).$$

All perturbed weights are required to remain positive, and the resulting weights are normalized so that their sum equals one.

The resulting perturbed weight vectors used in the sensitivity analysis are summarized in Table 6. The sensitivity of the CBCNFFPA and CBCNFFPG operators to changes in criterion weights is investigated using the weight scenarios presented in Table 6. These scenarios include increases and decreases in the importance of individual criteria, as well as simultaneous perturbations of multiple criteria, as shown in Table 7.

Table 6
Perturbed criterion weights

	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$	$\mathbb{C}_5$
Original weights	0.250	0.200	0.150	0.150	0.250
Increase $\mathbb{C}_1$	0.286	0.190	0.143	0.143	0.238
Decrease $\mathbb{C}_1$	0.211	0.211	0.158	0.158	0.263
Increase $\mathbb{C}_5$	0.238	0.190	0.143	0.143	0.286
Increase middle criteria	0.229	0.211	0.165	0.165	0.229
Mixed perturbation	0.300	0.170	0.170	0.130	0.230

Table 7

Sensitivity of CBCNFFPA and CBCNFFPG score values with respect to different criterion weight scenarios

Weight scenario	CBCNFFPA operator				
	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
Original weights	-0.421079	-0.371058	-0.456901	-0.449760	-0.335346
Increase $\mathbb{C}_1$	-0.420418	-0.370397	-0.456243	-0.449102	-0.334684
Decrease $\mathbb{C}_1$	-0.421810	-0.371790	-0.457628	-0.450487	-0.336077
Increase $\mathbb{C}_5$	-0.421775	-0.371753	-0.457601	-0.450460	-0.336041
Increase middle criteria	-0.421108	-0.371089	-0.456924	-0.449783	-0.335376
Mixed perturbation	-0.420154	-0.370133	-0.455978	-0.448837	-0.334420

Weight scenario	CBCNFFPG operator				
	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5
Original weights	-0.421317	-0.371177	-0.456968	-0.449933	-0.335414
Increase $\mathbb{C}_1$	-0.420662	-0.370518	-0.456311	-0.449279	-0.334755
Decrease $\mathbb{C}_1$	-0.422039	-0.371903	-0.457691	-0.450653	-0.336143
Increase $\mathbb{C}_5$	-0.422021	-0.371875	-0.457669	-0.450638	-0.336112
Increase middle criteria	-0.421332	-0.371200	-0.456986	-0.449945	-0.335441
Mixed perturbation	-0.420395	-0.370253	-0.456045	-0.449012	-0.334490

As can be seen from the table, changes in criterion weights result in relatively small deviations in the score values of alternatives in both the CBCNFFPA and CBCNFFPG operators. Nevertheless, relative preference order does not undergo any essential changes when taking into account the considered weight scenarios. Specifically, Ξ_5 still attains the highest score value while Ξ_3 continues to be the worst alternative. For the case of the CBCNFFPA operator, the aforementioned weight changes do not imply a shift in the overall ranking of the alternatives. Similar stability is also noted in the case of the CBCNFFPG operator. As such, it is evident that the proposed decision-making methodology demonstrates stability with respect to the considered criterion-weight perturbations analyzed in the current numerical example. At the same time, it should be noted that this finding can only be interpreted in the context of the sensitivity analysis being performed. Specifically, the absence of a ranking change under the particular weight perturbation scenarios does not mean that the proposed model is robust to any possible changes in the criterion importance weights. Indeed, larger or significantly different weight perturbations may lead to different aggregate values and thus different rankings. From an overall perspective, the weight sensitivity analysis enhances the parameter sensitivity analysis by confirming that the robustness of the framework does not depend on changes in the value of the Frank parameter α alone. From the analyzed range of α and the criterion weights, both CBCNFFPA and CBCNFFPG produce similar preference results for the considered example.

5.2.2 Sensitivity with Respect to CBCNF Evaluations

To examine the stability of the proposed decision-making model with respect to small changes in the input evaluations, a component-wise sensitivity analysis is performed on the CBCNF decision matrix. Since each CBCNFN contains positive and negative truth, indeterminacy, and falsity components together with a circular-radius component, the perturbation procedure is explicitly defined below to ensure reproducibility and preservation of CBCNFN admissibility.

For a CBCNFN $Z_\chi = (T^+, T^-, I^+, I^-, F^+, F^-, R)$, the perturbation is applied separately to the real and imaginary parts of the six truth, indeterminacy, and falsity components. Thus, the twelve scalar components

$$T_{\mathbb{R}}^+, T_{\mathbb{I}}^+, T_{\mathbb{R}}^-, T_{\mathbb{I}}^-, I_{\mathbb{R}}^+, I_{\mathbb{I}}^+, I_{\mathbb{R}}^-, I_{\mathbb{I}}^-, F_{\mathbb{R}}^+, F_{\mathbb{I}}^+, F_{\mathbb{R}}^-, F_{\mathbb{I}}^-$$

are perturbed, whereas the two circular-radius components $R_{\mathbb{R}}$ and $R_{\mathbb{I}}$ are kept unchanged. This choice allows the sensitivity analysis to assess the effect of perturbations in the preference-related

CBCNFN components without simultaneously changing the circular uncertainty radius. For a scalar component x , the positive and negative perturbations are defined as

$$x^{(+)} = \begin{cases} \min\{1, x + 0.01\}, & x \in [0, 1], \\ \min\{0, x + 0.01\}, & x \in [-1, 0], \end{cases}$$

and

$$x^{(-)} = \begin{cases} \max\{0, x - 0.01\}, & x \in [0, 1], \\ \max\{-1, x - 0.01\}, & x \in [-1, 0]. \end{cases}$$

Therefore, a positive perturbation moves a component upward by 0.01, whereas a negative perturbation moves it downward by 0.01. If a perturbation reaches a boundary, the value is clipped to the corresponding admissible interval. In particular, values equal to 0 or 1 for positive components and values equal to -1 or 0 for negative components remain admissible after perturbation.

The following perturbation scenarios are considered.

1. Positive perturbation ($+0.01$): all twelve T , I , and F real and imaginary components of every CBCNFN in the decision matrix are increased by 0.01, subject to the interval-preserving rule above.
2. Negative perturbation (-0.01): all twelve T , I , and F real and imaginary components of every CBCNFN in the decision matrix are decreased by 0.01, subject to the same admissibility-preserving rule.
3. Mixed perturbation (± 0.01): a deterministic mixed perturbation is introduced across the criteria. The components corresponding to C_1 , C_3 , and C_5 are increased by 0.01, whereas the components corresponding to C_2 and C_4 are decreased by 0.01. The same rule is applied to all twelve preference-related scalar components of each CBCNFN, while the circular radius components remain unchanged.

The perturbation procedure can therefore be represented by $\tilde{x}_{ij}^{(s)} = \Pi_{\mathcal{I}(x_{ij})} (x_{ij} + \delta_j^{(s)})$, where x_{ij} denotes any of the twelve perturbed scalar components of the CBCNFN corresponding to alternative i and criterion j ,

$$\delta_j^{(+)} = 0.01, \quad \delta_j^{(-)} = -0.01,$$

and, for the mixed scenario,

$$\delta_j^{(\pm)} = \begin{cases} +0.01, & j \in \{1, 3, 5\}, \\ -0.01, & j \in \{2, 4\}. \end{cases}$$

Here, $\Pi_{\mathcal{I}}$ denotes projection onto the admissible interval, with $\mathcal{I} = [0, 1]$ for positive components and $\mathcal{I} = [-1, 0]$ for negative components. The radius components are not modified.

For example, the original C_1 evaluation of alternative Ξ_1 is

$$Z_{\Xi_1, C_1} = (0.51 + i0.31, -0.71 - i0.21, 0.31 + i0.71, \\ -0.51 - i0.81, 0.21 + i0.91, -0.41 - i0.41, 0.81 + i0.51).$$

Under the $+0.01$ perturbation, the corresponding CBCNFN becomes

$$Z_{\Xi_1, C_1}^{(+)} = (0.52 + i0.32, -0.70 - i0.20, 0.32 + i0.72, \\ -0.50 - i0.80, 0.22 + i0.92, -0.40 - i0.40, 0.81 + i0.51).$$

Notice that the circular-radius component is unchanged. Similarly, under the -0.01 perturbation,

$$Z_{\Xi_1, C_1}^{(-)} = (0.50 + i0.30, -0.72 - i0.22, 0.30 + i0.70, \\ -0.52 - i0.82, 0.20 + i0.90, -0.42 - i0.42, 0.81 + i0.51).$$

These examples illustrate that the perturbation is applied to every preference-related component rather than to only a selected CBCNFN entry. For the mixed scenario, C_1 , C_3 , and C_5 receive a $+0.01$ perturbation, whereas C_2 and C_4 receive a -0.01 perturbation. Consequently, the mixed scenario

contains both positive and negative perturbations and is completely determined without any random selection of components. After constructing each perturbed decision matrix, the original criterion weights

$$\omega = (0.25, 0.20, 0.15, 0.15, 0.25)$$

and the Frank parameter $\alpha = 2$ are retained. The CBCNFFPA and CBCNFFPG operators are then applied exactly as in the main numerical example, and the score values are recalculated using the proposed score function. The resulting values are reported in Table 8. The purpose of this analysis is not to establish robustness against arbitrary changes in the input data, but to examine the local stability of the proposed method under the explicitly specified perturbations of magnitude 0.01. Hence, the conclusions are restricted to the perturbation magnitude and scenarios considered here.

Table 8
Sensitivity analysis of the score values under different perturbation scenarios

Scenario	Alternative	CBCNFFPA	CBCNFFPG
+0.01	Ξ_1	-0.418576	-0.418797
+0.01	Ξ_2	-0.368558	-0.368660
+0.01	Ξ_3	-0.454368	-0.454445
+0.01	Ξ_4	-0.447242	-0.447389
+0.01	Ξ_5	-0.332844	-0.332904
-0.01	Ξ_1	-0.424296	-0.424483
-0.01	Ξ_2	-0.374278	-0.374378
-0.01	Ξ_3	-0.460121	-0.460152
-0.01	Ξ_4	-0.452959	-0.453105
-0.01	Ξ_5	-0.338567	-0.338624
± 0.01	Ξ_1	-0.420843	-0.421164
± 0.01	Ξ_2	-0.370833	-0.370983
± 0.01	Ξ_3	-0.456697	-0.456799
± 0.01	Ξ_4	-0.449511	-0.449729
± 0.01	Ξ_5	-0.335117	-0.335205

Table 8 presents the score values obtained under the positive, negative, and mixed ± 0.01 perturbation scenarios. For the CBCNFFPA operator, the preference order remains $\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$ in all three scenarios. Similarly, for the CBCNFFPG operator, the preference order remains $\Xi_5 > \Xi_2 > \Xi_1 > \Xi_4 > \Xi_3$ throughout the sensitivity analysis. Thus, the specified ± 0.01 perturbations of the CBCNFN evaluations do not alter the ranking within either aggregation operator. In particular, Ξ_5 remains the highest-ranked alternative and Ξ_3 remains the lowest-ranked alternative under all considered perturbation scenarios.

6. Theoretical Comparison with Recent Parametrized Aggregation Frameworks

To further understand the properties of the newly proposed CBCNFFPA and CBCNFFPG operators, a comparison is performed between the new operators and recent fuzzy aggregation approaches based on parameters. However, the aim of the comparison is not to directly apply all existing operators within the CBCNFS setting, because the mathematical domains of the respective operators differ. Instead, the comparison is done with respect to the domain of information, aggregation method, parameterization, criteria weighting, and consideration of similarity interaction of aggregated information. The proposed operators are compared with recently developed aggregation approaches based on Sugeno–Weber, Aczel–Alsina, circular spherical fuzzy, complex T-spherical Hamacher, and spherical fuzzy rough Schweizer–Sklar frameworks in Table 9.

Table 9

Theoretical comparison of the proposed CBCNFFPA and CBCNFFPG operators with recent aggregation operators

Approach	Information Domain	Parameterized	Complex	Bipolar	Circular
Sugeno-Weber IVT-spherical fuzzy model [28]	IVT-spherical fuzzy	Yes	No	No	No
Aczel-Alsina spherical fuzzy model [29]	Spherical fuzzy	Yes	No	No	No
Circular spherical fuzzy model [30]	Circular spherical fuzzy	Yes/Model-dependent	No	No	Yes
Interval-valued complex T-spherical Hamacher model [31]	IV complex T-spherical fuzzy	Yes	Yes	No	No
Spherical fuzzy rough Schweizer-Sklar model [32]	Spherical fuzzy rough	Yes	No	No	No
Proposed CBC-NFFPA/CBCNFFPG	CBCNFS	Yes	Yes	Yes	Yes

Approach	Criterion Weights	T-norm/T-conorm Family	Aggregation Type	Computational Complexity
Sugeno-Weber IVT-spherical fuzzy model [28]	SWARA	Sugeno-Weber	Sugeno-Weber	Relatively lower
Aczel-Alsina spherical fuzzy model [29]	AHP	Aczel-Alsina	Aczel-Alsina	Relatively lower
Circular spherical fuzzy model [30]	Model-dependent	Model-dependent	Circular fuzzy aggregation	Moderate
Interval-valued complex T-spherical Hamacher model [31]	Model-dependent	Hamacher	Hamacher	Relatively higher
Spherical fuzzy rough Schweizer-Sklar model [32]	Model-dependent	Schweizer–Sklar	Schweizer–Sklar	Moderate
Proposed CBC-NFFPA/CBCNFFPG	Yes	Frank	Frank Power	Relatively higher

Comparison in Table 9 reveals that CBCNFFPA and CBCNFFPG operators are different from other recently proposed aggregation frameworks in terms of information richness and similarity-based Power weighting method. Parametrized aggregation methods of Sugeno-Weber and Aczel-Alsina frameworks may be advantageous when the uncertainty can be modeled using T-spherical or spherical fuzzy information. A simpler information structure may lead to reduced computational effort and facilitate the collection of expert opinions. The T-norm/T-conorm class is mentioned at the level of the aggregation framework when it is explicitly described in terms of that approach. The computational complexity is a qualitative measure depending on the richness of the structure used to represent and aggregate the information; it is not an asymptotic complexity analysis. The terms “relatively low,” “moderate,” and “relatively high” refer to the qualitative computational complexity associated with the respective framework.

The circular spherical fuzzy approach is particularly relevant to the proposed framework because both approaches incorporate circular uncertainty. Nevertheless, the proposed CBCNFS approach can model both positive and negative information, complex-valued components, and independent T, I, and F information. Thus, the proposed model provides a richer representational capacity when bipolar, complex, neutrosophic, and circular uncertainties occur simultaneously.

The interval-valued complex T-spherical Hamacher approach is related to the proposed framework because both approaches use complex-valued fuzzy information and parameterized aggregation operators. Nevertheless, the proposed CBCNFS approach also accounts for bipolarity and circular information. Likewise, the spherical fuzzy rough Schweizer-Sklar approach provides a flexible parametrized aggregation operator and will be suitable for dealing with rough uncertainty, but it does not simultaneously address complex-valued, bipolar, circular, and neutrosophic uncertainties.

Another feature of the proposed CBCNFFPA and CBCNFFPG operators is that they utilize the Power mechanism. In contrast to aggregation operators that apply only pre-defined criterion weights, the Power mechanism modifies the relative influence of the evaluations being aggregated in line with their mutual similarity. Evaluations that receive stronger support from other evaluations have greater influence on the aggregation result. However, this mechanism is not to be regarded as a causal interaction between the criteria, but rather as a similarity-based consensus mechanism for the aggregated evaluations.

The additional flexibility and sophisticated information structure provided by the proposed framework and its Power mechanism, however, are accompanied by a higher level of mathematical complexity. More components need to be elicited from the decision makers, and more distance and support measurements are required. Thus, the proposed CBCNFFPA and CBCNFFPG operators are more suitable when the decision problem involves bipolar, complex, neutrosophic and circular information in the problem under consideration. If this kind of uncertainty structure is not needed, a simpler parametrized aggregation procedure can be more appropriate.

7. Conclusion

The proposed MADM framework is based on CBCNFSs and Frank power aggregation operators. The proposed framework makes use of the circular bipolar complex neutrosophic fuzzy power average and power geometric aggregation operators to aggregate and rank alternatives based on multidimensional information. The major benefit of the proposed framework is its ability to represent multidimensional information within a unified structure. In particular, the CBCNFS model enables us to represent positive and negative information, T, I, F, real and imaginary parts of a complex number, and circular information in a single structure. Moreover, Frank operators provide parameterized behaviour of the aggregation process, while the Power aggregation process considers the mutual influence among the aggregated information. The comparison indicates that the proposed framework can represent a broader range of information characteristics than some existing fuzzy-set frameworks. However, greater representational capacity does not by itself establish better decision-making performance or guarantee superior rankings. In addition, the results of sensitivity analysis reveal robustness in terms of particular cases under consideration but do not indicate robustness in general decision-making cases. The simultaneous use of bipolar, complex, neutrosophic, and circular information, together with Frank Power aggregation, is meaningful only when these features are relevant to the decision problem. For instance, bipolar information is relevant when both positive and negative information must be considered; complex-valued information is relevant when evaluations contain real and imaginary components; neutrosophic information is relevant when truth, indeterminacy, and falsity need to be distinguished; and circular information is relevant when a radius component is required to represent circular uncertainty.

Future research may extend the proposed framework through dynamic parameter adjustment, hybrid aggregation strategies, and machine-learning-based approaches for determining criterion weights. In addition, validation using real-world decision data and structured evaluations obtained from domain experts or users would provide further evidence regarding the practical effectiveness of

the proposed approach.

7.1 Advantages of the Proposed Approach

The main benefits of the introduced CBCNFS model and the CBCNFFPA and CBCNFFPG operators can be stated as follows:

- First, the CBCNFS model offers an extensive structure for information representation because it can incorporate positive and negative information, T, I, and F degrees, real and imaginary parts, as well as circular information via the radius parameter. Consequently, the information can be represented in different forms within one framework.
- Second, the use of the Frank power aggregation operator considers the support among the values that are aggregated and decreases the impact of an evaluation that has a relatively low level of support by other evaluations. In other words, the power operator allows an adaptive weight to be assigned.
- Finally, the introduced CBCNFFPA and CBCNFFPG operators support arithmetic and geometric aggregation, respectively. Therefore, the framework can aggregate CBCNF information in a flexible manner.

Thus, the main benefit of the introduced framework is not an increase in decision-making performance but a higher representational capacity. Such a framework is especially useful when the information provided by decision makers has bipolar, complex, neutrosophic, and circular characteristics.

7.2 Limitation and Future Applicability of the Proposed Approach

Although the proposed CBCNFFPA and CBCNFFPG approaches provide extensive representation capabilities, they have some limitations. First, using all of the aforementioned number types simultaneously requires decision-makers to provide a relatively large amount of information. The process of evaluation may include several parameters, which complicates the process of data collection and increases the workload of decision-makers. Second, the mathematical description of the proposed model is more complex than that of some existing fuzzy aggregation models. Third, the Power aggregation process depends on the chosen distance and support measures and is therefore sensitive to the value of the operational parameter α . Thus, the resulting aggregation behaviour and rankings may depend on these modelling choices. Consequently, the developed framework becomes more relevant in those cases when bipolar, complex-valued, neutrosophic, and circular characteristics are genuinely present and have to be handled in combination with parameterized Frank aggregation and support-aware Power aggregation. If this is not the case, then the application of the whole CBCNFS framework could lead to unnecessary modelling complexity. In this case, it might become more appropriate to use a simpler fuzzy model, which would be easier to interpret and apply.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this study.

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