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Cycle-Super Magic Labeling of Polyomino Linear and Zig-Zag Chains

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ABSTRACT

A bijective function with domain union of vertex and edge set to a range natural numbers to onward count of vertices and edges of a graph. If there is a bijective function G, then G is called as a H-magic graph, along with the condition that every subgraph H' from the original graph G, H' is isomorphic to H and with the interesting fact that sum of all the values of vertices and edges is constant for all H'. This condition can be more advance and strict when the first order-numbers assigned to the vertices only and then this definitions is called as H-super magic. In this paper, we study some polyomino structures, including zig-zag and linear chains. We studied, C8-super magic labeling of zig-zag, linear chains and also the disjoint union of non-isomorphic copies of both chains.

Introduction

Let a graph having properties with, finiteness, loop-less and without any multiple edges and direction, say $G = (V, E)$, in this V and E are its vertex and edge sets respectively. A labeling is a function with domain of a graph's vertex and/or edge set and range usually from the positive integers. A graph G satisfies a H-covering, if all of the edges in the edge set of a graph G lies as to be a subgraph of G and also isomorphic to H.

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Let a bijective function $\mathcal{G}: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$, if there is a bijective function $\mathcal{G}$. then G is called as a H-magic graph, along with the condition that every subgraph H' from the original graph G, H' is isomorphic to H and with the interesting fact that $\sum_{v \in V} \mathcal{G}(v) + \sum_{e \in E} \mathcal{G}(e)$, is constant for all H'. This condition can be more advance and strict when the first $|V(G)|$-numbers assigned to vertices only and then this definitions is called as H-super magic.

The complete bipartite $K_{n,m}$ and star graph $K_{1,n}$, for some h are $K_{1,h}$-super magic graphs. These classes are the first families of graphs studied very first with the introduction of super magic labeling by [1]. The path and cycle graph are P_h-super magic proved in the seminal work. Some basic and graphs obtained by some operation, like wheel, prism, book and windmill graphs are cycle-super magic graphs proved in [2] and devised the term for cycle magic and super magic labeling. Moreover, this definition is also implemented on subdivided wheel graph in the same work.

The graph constructed by cycle and path such as ladders, book and fans graphs are studied by [3], under the definition of cycle-super magic. The disjoint union of K-isomorphic copies of G is a G-super magic graph, proved by [4]. In [5, 6], proved some results on the edge super magic labeling and also studied about the deficiency of graphs. About the deficiency of graphs in terms of edge super magic are available in [7]. The uniform subdivision of wheel graph and super edge-magic labeling of some chemical graphs are discussed in [8, 9]. The more detail of literature related to this topic and conceptual study we refer to the very broad and dynamical survey [10].

Recently some basic sheets and lattice graphs along with general pumpkin graph are studied in [11], under the definition of super edge magic labeling. Another recent work from [12], they studied the H-super magic labeling of two basic family of graphs, which are cycle and path. The similar definition is studied in [13] for some generalized class of graphs. Some advanced research on this topic and similar definitions are studied in [14–16].

A system of $\mathcal{N}$-polyomino is a finite planar graph that each interior portion is called as cell which is associated with regular $4\mathcal{N}$-cycle. The number of $4\mathcal{N}$-cycles in a fragment S is called its length and symbolized by $l(S)$. For any $n \geq 2$ $4\mathcal{N}$-cycles one has $2 \leq l(S) \leq n$. Particularly, a $\mathcal{N}$-polyomino chain is a linear chain iff it contains exactly one fragment. A $\mathcal{N}$-polyomino chain is a zig-zag chain iff the length of each fragment is 2. In Figure 1 and 2 zig-zag and linear polyomino chains are shown respectively. For recent research work graph theoretical parameters one can see the citations [17–30].

Main Results

For $\mathcal{N} \geq 3$, let $G \cong \mathcal{ZP}_{\mathcal{N}-1}$ be a zig-zag polyomino network the vertex and edge sets of graph G as follows:

$$\begin{aligned}
\mathbf{V}(\mathbf{G}) = &\{s_a,\ t_a : 1 \leq a \leq \mathcal{N}\} \cup \left\{s_c^b,\ t_c^b : 1 \leq b \leq \mathcal{N}-1, 1 \leq c \leq 4\right\}, \\
\mathbf{E}(\mathbf{G}) = &\{s_a t_a : 1 \leq a \leq \mathcal{N}\} \cup \{s_a s_{a+1},\ t_a t_{a+1} : 1 \leq a \leq \mathcal{N}-1\} \cup \{s_a s_c^b : c = 1, 1 \leq b \leq \\
&\mathcal{N}-1, 1 \leq a \leq \mathcal{N}-1\} \cup \left\{s_c^b s_a : c = 4, 1 \leq b \leq \mathcal{N}-1, 1 \leq a \leq \mathcal{N}-1\right\} \cup \{t_a t_c^b : \\
&c = 1, 1 \leq b \leq \mathcal{N}-1, 1 \leq a \leq \mathcal{N}-1\} \cup \{t_c^b t_a : c = 4, 1 \leq b \leq \mathcal{N}-1, 3 \leq a \leq \\
&\mathcal{N}-1\} \cup \left\{s_c^b s_{c+1}^b,\ t_c^b t_{c+1}^b : 1 \leq c \leq 3, 1 \leq b \leq \mathcal{N}-1\right\}. \qquad (1)
\end{aligned}$$

Theorem 0.1. *For any positive integer $\mathcal{N} \geq 3$, the graph $G \cong \mathcal{ZP}_{\mathcal{N}-1}$ is C_8-supermagic.*

Proof. Consider $\mathfrak{o} = |V(G)|$ and $\mathfrak{s} = |E(G)|$. then $\mathfrak{o} = 2\mathcal{N} + 4\left(\lfloor \frac{\mathcal{N}}{2} \rfloor + \lceil \frac{\mathcal{N}}{2} - 1 \rceil\right)$, $\mathfrak{s} = \mathcal{N} + 2(\mathcal{N}-1) + 4\left(\lfloor \frac{\mathcal{N}}{2} \rfloor + \lceil \frac{\mathcal{N}}{2} - 1 \rceil\right)$.

Describe a total labeling $\mathcal{G} : V(G) \cup E(G) \to \{1, 2, \ldots, 5\mathcal{N} - 2 + 4\left(\left\lfloor \frac{\mathcal{N}}{2} \right\rfloor + \left\lceil \frac{\mathcal{N}}{2} - 1 \right\rceil\right)\}$, as follows:

Case 1: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}(s_a) = a. \tag{2}$$

Case 2: For $1 \leq b \leq \mathcal{N} - 1$, $1 \leq c \leq 4$, $b = \text{odd}$;

$$\mathcal{G}\left(s_c^b\right) = b + 1 + c(\mathcal{N} - 1). \tag{3}$$

Case 3: For $2 \leq a \leq \mathcal{N} - 1,\ a = \text{even}$;

$$\mathcal{G}(s_a s_{a+1}) = 8\mathcal{N} + a - 5. \tag{4}$$

Case 4: For $1 \leq b \leq \mathcal{N} - 1,\ 1 \leq c \leq 3,\ b = \text{odd}$;

$$\mathcal{G}\left(s_c^b s_{c+1}^b\right) = 14\mathcal{N} - b - c(\mathcal{N} - 1) - 10. \tag{5}$$

Case 5: For $1 \leq b \leq \mathcal{N} - 1,\ 1 \leq a \leq \mathcal{N} - 1,\ c = 1$, a and b are odd;

$$\mathcal{G}\left(s_a s_c^b\right) = 8\mathcal{N} - 4 - c + b. \tag{6}$$

Case 6: $c = 4,\ b = \text{odd},\ a = \text{even}$;

$$\mathcal{G}\left(s_c^b s_a\right) = 10\mathcal{N} - b + c - 10. \tag{7}$$

Case 7: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}(t_a) = 6\mathcal{N} - a - 3. \tag{8}$$

Case 8: For $2 \leq b \leq \mathcal{N} - 1,\ 1 \leq c \leq 4,\ b = \text{even}$;

$$\mathcal{G}\left(t_c^b\right) = b + 1 + c(\mathcal{N} - 1). \tag{9}$$

Case 9: For $1 \leq a \leq \mathcal{N} - 1,\ a = \text{odd}$;

$$\mathcal{G}(t_a t_{a+1}) = 7\mathcal{N} - 4 + a. \tag{10}$$

Case 10: For $2 \leq b \leq \mathcal{N} - 1,\ 1 \leq c \leq 3,\ b = \text{even}$;

$$\mathcal{G}\left(t_c^b t_{c+1}^b\right) = 14\mathcal{N} - b - c(\mathcal{N} - 1) - 10. \tag{11}$$

Case 11: For $2 \leq b \leq \mathcal{N} - 1,\ 2 \leq a \leq \mathcal{N} - 1,\ c = 1$, a and b are even;

$$\mathcal{G}\left(t_a t_c^b\right) = 7\mathcal{N} + b - c - 3. \tag{12}$$

Case 12: For $2 \leq b \leq \mathcal{N} - 1,\ 3 \leq a \leq \mathcal{N} - 1,\ c = 4,\ b = \text{even},\ a = \text{odd}$;

$$\mathcal{G}\left(t_c^b t_a\right) = 10\mathcal{N} - b + c - 10. \tag{13}$$

Case 13: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}(s_a t_a) = 7\mathcal{N} - a - 3. \tag{14}$$

It is simple to see that for every subcycle $C_8^d : 1 \leq d \leq (\mathcal{N} - 1)$ of $\mathcal{ZP}_{\mathcal{N}-1}$, the weight of C_8^d is $97\mathcal{N} - 58$. Hence $\mathcal{ZP}_{\mathcal{N}-1}$ is C_8-supermagic. □

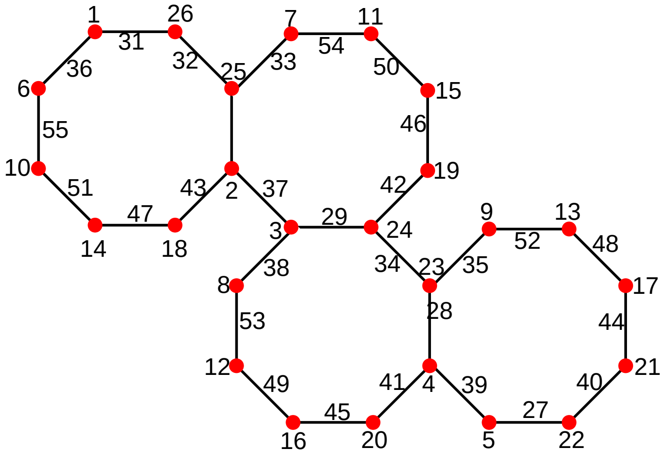

Figure 1: C_8-super magic labeling of $\mathcal{ZP}_4$ with each cell weight is 427.

Consider $\mathcal{N} \geq 2, G \cong \mathcal{LP}_{\mathcal{N}-1}$be a linear $\mathcal{N}$-polymino chain graph and let $\mathfrak{o} = |V(G)|$ and $\mathfrak{s} = |E(G)|$. Then $\mathfrak{o} = 6\mathcal{N} - 4$, $\mathfrak{s} = 7\mathcal{N} - 6$, the vertex and edge sets of graph G as follow:

$$\begin{aligned}
\mathbf{V}(\mathbf{G}) &= \{s_a,\ t_a : 1 \leq a \leq \mathcal{N}\} \cup \{s_b^c,\ t_b^c : 1 \leq c \leq \mathcal{N} - 1, b = 1, 2\}, \\
\mathbf{E}(\mathbf{G}) &= \{s_a t_a : 1 \leq a \leq \mathcal{N}\} \cup \{s_a s_b^c : 1 \leq c \leq \mathcal{N} - 1, b = 1, 1 \leq a \leq \mathcal{N} - 1\} \\
&\quad \cup \{t_a t_b^c : 1 \leq c \leq \mathcal{N} - 1, b = 1, 1 \leq a \leq \mathcal{N} - 1\} \cup \{s_b^c s_a,\ t_b^c t_a : b = 2, \\
&\quad 2 \leq a \leq \mathcal{N} - 1, 1 \leq c \leq \mathcal{N} - 1\} \cup \left\{s_b^c s_{b+1}^c : b = 1, 1 \leq c \leq \mathcal{N} - 1\right\} \\
&\quad \cup \left\{t_b^c t_{b+1}^c : b = 1, 1 \leq c \leq \mathcal{N} - 1\right\}.
\end{aligned} \tag{15}$$

Theorem 0.2. *The graph $G \cong \mathcal{LP}_{\mathcal{N}-1}$, where $\mathcal{N} \geq 2$ is C_8-supermagic.*

Proof. Describe a total labeling $\mathcal{G} : V(G) \cup E(G) \to \{1, 2, \ldots, 13\mathcal{N} - 10\}$. as follows:

Throughout the following labeling, we will consider $1 \leq c \leq \mathcal{N} - 1$

Case 1: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}(s_a) = a. \tag{16}$$

Case 2: For $b = 1, 2$;

$$\mathcal{G}(s_b^c) = 2\mathcal{N} + 2(c - 1) + 1 + (b - 1). \tag{17}$$

Case 3: For $b = 1,\ 1 \leq a \leq \mathcal{N} - 1$;

$$\mathcal{G}(s_a s_b^c) = 3\mathcal{N} + 5(\mathcal{N} - b) + c, \tag{18}$$

$$\mathcal{G}\left(s_b^c s_{b+1}^c\right) = 3\mathcal{N} + 10(\mathcal{N} - 1) - 2(c - b). \tag{19}$$

Case 4: For $b = 2,\ 2 \leq a \leq \mathcal{N}$;

$$\mathcal{G}(s_b^c s_a) = 3\mathcal{N} + 10(\mathcal{N} - 1) - 2(c - 1) - (b - 1). \tag{20}$$

Case 5: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}(t_a) = 2\mathcal{N} - (a-1). \tag{21}$$

Case 6: For $b = 1, 2$;

$$\mathcal{G}(t_b^c) = 2\mathcal{N} + 2(\mathcal{N}-1) + 2(c-1) + (b-1) + 1. \tag{22}$$

Case 7: For $b = 1,\ 1 \leq a \leq \mathcal{N} - 1$;

$$\mathcal{G}(t_a t_b^c) = 2\mathcal{N} + 5(\mathcal{N}-b) + (c-1) + (b+1), \tag{23}$$

$$\mathcal{G}\left(t_b^c t_{b+1}^c\right) = 3\mathcal{N} + 8(\mathcal{N}-1) - 2(c-b). \tag{24}$$

Case 8: For $b = 2,\ 2 \leq a \leq \mathcal{N}$;

$$\mathcal{G}(t_b^c t_a) = 3\mathcal{N} + 8(\mathcal{N}-1) - 2(c-1) - (b-1). \tag{25}$$

Case 9: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}(s_a t_a) = 7\mathcal{N} - a - 3. \tag{26}$$

After labeling the vertices and edges of graph G, we can see that the every subcycle $C_8^d : 1 \leq \ell \leq \mathcal{N} - 1$ under the labeling $\mathcal{G}$, the weight of every cycle C_8^d is $93\mathcal{N} - 50$. Hence the graph $G \cong \mathcal{LP}_{\mathcal{N}-1}$ is C_8-supermagic. □

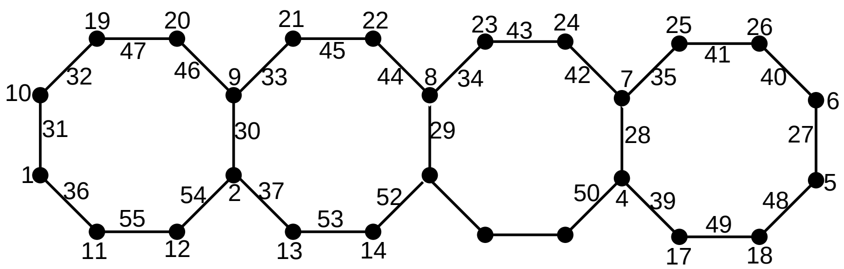

Figure 2: C_8super magic labeling of $\mathcal{LP}_4$ with each cell weight is 415.

In the following theorem, we consider C_8-supermagic labeling for disjoint union of non-isomorphic copies of $\mathcal{N}$-polyomino linear chain.

Theorem 0.3. *For some positive integers α, β and $\mathcal{N} \geq 3$, the graph $G \cong \alpha\mathcal{LP}_n \cup \beta\mathcal{LP}_{\mathcal{R}}$, where $n = \mathcal{N} - 1, Re = \mathcal{N} - i$ and $1 \leq i \leq \mathcal{N} - 2$ is C_8-supermagic.*

Proof. Consider $\mathfrak{o} = |V(G)|$ and $\mathfrak{s} = |E(G)|$. Then $\mathfrak{o} = 2[\alpha(3\mathcal{N}-2) + \beta(3\mathcal{R}-2)], \mathfrak{s} = \alpha(7\mathcal{N}-6) + \beta(7\mathcal{R}-6)$.

The vertex and edge sets of graph G as follows:

$$
\begin{aligned}
\mathbf{V}(\mathbf{G}) = & \left\{s_a^b : 1 \le a \le \mathcal{N}, 1 \le b \le \alpha\right\} \cup \left\{t_a^b : 1 \le a \le \mathcal{N}, 1 \le b \le \alpha\right\} \\
& \cup \left\{u_a^c : 1 \le a \le \mathcal{R}, 1 \le c \le \beta\right\} \cup \left\{v_a^c : 1 \le a \le \mathcal{R}, 1 \le c \le \beta\right\} \\
& \cup \left\{w_{d,e}^h : 1 \le e \le \mathcal{N}-1, 1 \le h \le \alpha, d = 1,2\right\} \\
& \cup \left\{x_{d,e}^h : 1 \le e \le \mathcal{N}-1, 1 \le h \le \alpha, d = 1,2\right\} \\
& \cup \left\{y_{d,e}^h : d = 1,2, 1 \le e \le \mathcal{R}-1, 1 \le h \le \beta\right\} \\
& \cup \left\{z_{d,e}^h : d = 1,2, 1 \le e \le \mathcal{R}-1, 1 \le h \le \beta\right\}, \\
\mathbf{E}(\mathbf{G}) = & \left\{s_a^b t_a^b : 1 \le a \le \mathcal{R}, 1 \le b \le \alpha\right\} \cup \left\{s_a^b t_a^b : \mathcal{R}+1 \le a \le \mathcal{N}, 1 \le b \le \alpha\right\} \\
& \cup \left\{u_a^c v_a^c : 1 \le c \le \beta, 1 \le a \le \mathcal{R}-1\right\} \cup \left\{u_a^c v_a^c : 1 \le c \le \beta, a = \mathcal{R}\right\} \\
& \cup \left\{s_a^b w_{d,e}^h : 1 \le h = b \le \alpha, 1 \le a = e \le \mathcal{R}-1, d = 1\right\} \\
& \cup \left\{s_a^b w_{d,e}^h : 1 \le h = b \le \alpha, \mathcal{R} \le a = e \le \mathcal{N}-1, d = 1\right\} \\
& \cup \left\{t_a^b x_{d,e}^h : 1 \le h = b \le \alpha, 1 \le a = e \le \mathcal{R}, d = 1\right\} \\
& \cup \left\{t_a^b x_{d,e}^h : 1 \le h = b \le \alpha, \mathcal{R}-1 \le a = e \le \mathcal{R}, d = 1\right\} \\
& \cup \left\{u_a^c y_{d,g}^f : 1 \le c = f \le \beta, 1 \le a = g \le \mathcal{R}-2, d = 1\right\} \\
& \cup \left\{u_a^c y_{d,g}^f : 1 \le c = f \le \beta, a = g = \mathcal{R}-1, d = 1\right\} \\
& \cup \left\{v_a^c z_{d,g}^f : 1 \le c = f \le \beta, 1 \le a = g \le \mathcal{R}-1, d = 1\right\} \\
& \cup \left\{w_{d,e}^h w_{d+1,e}^h : 1 \le h \le \alpha, 1 \le e \le (\mathcal{N}-1), d = 1\right\} \\
& \cup \left\{w_{d,e}^h s_a^b : 1 \le h = b \le \alpha, 2 \le a \le \mathcal{N}, d = 2, e+1 = aanv1 \le e \le \mathcal{N}-1\right\} \\
& \cup \left\{x_{d,e}^h x_{d+1,e}^h : 1 \le h \le \alpha, 1 \le e \le \mathcal{N}-1, d = 1\right\} \\
& \cup \left\{x_{d,e}^h t_a^b : 1 \le h = b \le \alpha, 2 \le a \le \mathcal{N}, d = 2, e+1 = a, \text{ and, } 1 \le e \le \mathcal{N}-1\right\} \\
& \cup \left\{y_{d,g}^f y_{d+1,g}^f : 1 \le f \le \beta, 1 \le g \le \mathcal{R}-1, d = 1\right\} \\
& \cup \left\{y_{d,g}^f u_a^c : 1 \le f = c \le \beta, 2 \le a \le \mathcal{R}, d = 2, g+1 = a, \text{ and, } 1 \le g \le \mathcal{R}-1\right\} \\
& \cup \left\{z_{d,g}^f z_{d+1,g}^f : 1 \le f \le \beta, 1 \le g \le \mathcal{R}-1, d = 1\right\} \\
& \cup \left\{z_{d,g}^f v_a^c : 1 \le f = c \le \beta, 2 \le a \le \mathcal{R}-1, d = 2, g+1 = a, \text{ and, } 1 \le g \le \mathcal{R}-1\right\}.
\end{aligned} \tag{27}
$$

We will describe a total labeling $\mathcal{G} : V(G) \cup E(G) \to \{1, 2, \ldots, \alpha(13\mathcal{N}-10) + \beta(13\mathcal{R}-10)\}$.

Throughout the following labeling, we will consider $1 \le b \le \alpha,\ 1 \le c \le \beta,\ 1 \le h \le \alpha,\ 1 \le f \le \beta$.

Case 1: For $1 \le a \le \mathcal{N}$;

$$\mathcal{G}\left(s_a^b\right) = a + \mathcal{N}(b-1). \tag{28}$$

Case 2: For $1 \le e \le \mathcal{N}-1,\ d = 1,2$;

$$\mathcal{G}\left(w_{d,e}^h\right) = 2\alpha\mathcal{N} + 2\beta\mathcal{R} + h + \alpha(d-1) + 2\alpha(e-1). \tag{29}$$

Case 3: For $d = 1$;
with $1 \le e \le \mathcal{R}-1$,$1 \le a \le \mathcal{R}-1$;

$$\mathcal{G}\left(s_a^b w_{d,e}^h\right) = \mathfrak{o} + \alpha\mathcal{N} + \beta\mathcal{R} + \alpha(\mathcal{N}-1) + \beta(\mathcal{R}-1) + d + (\alpha+\beta)(a-1) + (b-1), \tag{30}$$

with $\mathcal{R} \leq a \leq \mathcal{N}-1, \mathcal{R} \leq e \leq \mathcal{N}-1$;

$$\mathcal{G}\left(s_a^b w_{d,e}^h\right) = \mathfrak{o} + \alpha\mathcal{N} + \beta\mathcal{R} + \alpha\left(\mathcal{N}-1\right) + \beta\left(\mathcal{R}-1\right) + d + \left(\alpha+\beta\right)\left(\mathcal{R}-2\right) + \alpha + \alpha\left(a-\mathcal{R}\right) + \left(b-1\right), \tag{31}$$

with $1 \leq e \leq \mathcal{N}-1$

$$\mathcal{G}\left(w_{d,e}^h w_{d+1,e}^h\right) = \mathfrak{o} + \mathfrak{s} - 2\alpha\left(e-1\right) - \left(h-d\right). \tag{32}$$

Case 4: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}\left(t_a^b\right) = 2\alpha\mathcal{N} + 2\beta\mathcal{R} - \mathcal{N}\left(b-1\right) - a + 1. \tag{33}$$

Case 5: For $1 \leq a \leq \mathcal{R}$,$d = 1$,$1 \leq e \leq \mathcal{R}$;

$$\mathcal{G}\left(t_a^b x_{d,e}^h\right) = \mathfrak{o} + \alpha\mathcal{N} + \beta\mathcal{R} + d + \left(\alpha+\beta\right)\left(a-1\right) + \left(b-1\right). \tag{34}$$

Case 6: For $1 \leq e \leq \mathcal{N}-1$,$d = 1, 2$;

$$\mathcal{G}\left(x_{d,e}^h\right) = 2\alpha\mathcal{N} + 2\beta\mathcal{R} + h + \alpha\left(d-1\right) + 2\alpha\left(e-1\right) + 2\alpha\left(\mathcal{N}-1\right). \tag{35}$$

Case 7: For $d = 1$;
with $\mathcal{R}+1 \leq a \leq \mathcal{N}-1, \mathcal{R}+1 \leq e \leq \mathcal{N}-1$;

$$\mathcal{G}\left(t_a^b x_{d,e}^h\right) = \mathfrak{o} + \alpha\mathcal{N} + \beta\mathcal{R} + \mathcal{R}\left(\alpha+\beta\right) + \alpha[a-\left(\mathcal{R}+1\right)] + \left(b-1\right) - \left(\beta-1\right), \tag{36}$$

with $1 \leq e \leq \mathcal{N}-1$;

$$\mathcal{G}\left(x_{d,e}^h x_{d+1,e}^h\right) = \mathfrak{o} + \mathfrak{s} - 2\alpha\left(\mathcal{N}-1\right) - 2\alpha\left(e-1\right) - \left(h-d\right). \tag{37}$$

Case 8: For $d = 2$, $2 \leq a \leq \mathcal{N}, e+1 = a,$ and, $1 \leq e \leq \mathcal{N}-1$;

$$\mathcal{G}\left(x_{d,e}^h t_a^b\right) = \mathfrak{o} + \mathfrak{s} - 2\alpha\left(e-1\right) - 2\alpha\left(\mathcal{N}-1\right) - \left(h-d+1\right) - \alpha. \tag{38}$$

Case 9: For $1 \leq a \leq \mathcal{R}$;

$$\mathcal{G}\left(s_a^b t_a^b\right) = \mathfrak{o} + \alpha\mathcal{N} + \beta\mathcal{R} - \left(a-1\right)\left(\alpha+\beta\right) - \left(b-1\right). \tag{39}$$

Case 10: For $\mathcal{R}+1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}\left(a_a^b t_a^b\right) = \mathfrak{o} + \alpha\left(\mathcal{N}-\mathcal{R}\right) + \beta - \alpha[a-\left(\mathcal{R}+1\right)] - \left(b-1\right). \tag{40}$$

Case 11: For $1 \leq a \leq \mathcal{R}$;

$$\mathcal{G}\left(u_a^c\right) = \alpha\mathcal{N} + a + \mathcal{R}\left(c-1\right). \tag{41}$$

Case 12: For $1 \leq g \leq \mathcal{R}-1$,$d = 1, 2$;

$$\mathcal{G}\left(y_{d,g}^f\right) = 2\alpha\mathcal{N} + 2\beta\mathcal{R} + f + \beta\left(d-1\right) + 2\beta\left(g-1\right) + 4\alpha\left(\mathcal{N}-1\right). \tag{42}$$

Case 13: For $d = 1$;
with $1 \leq g \leq \mathcal{R}-1$;

$$\mathcal{G}\left(y_{d,g}^f y_{d+1,g}^f\right) = \mathfrak{o} + \mathfrak{s} - 4\alpha\left(\mathcal{N}-1\right) - 2\alpha\left(g-1\right) - \left(f-d\right), \tag{43}$$

with $1 \leq a \leq \mathcal{R}-2, 1 \leq g \leq \mathcal{R}-2$;

$$\mathcal{G}\left(u_a^c y_{d,g}^f\right) = \mathfrak{o} + \alpha\mathcal{N} + \beta\mathcal{R} + \alpha(\mathcal{N}-1) + \beta(\mathcal{R}-1) + d + \alpha + 2\beta(a-1) + (c-1), \tag{44}$$

with $a = \mathcal{R}-1, g = \mathcal{R}-1$;

$$\mathcal{G}\left(u_a^c y_{d,g}^f\right) = \mathfrak{o} + \mathfrak{s} - (\beta-1) + (c-1) - 4\alpha(\mathcal{N}-1) - 4\beta(\mathcal{R}-1). \tag{45}$$

Case 14: For $d = 2$, with $2 \leq a \leq \mathcal{R}, g+1 = a$ and $1 \leq g \leq \mathcal{R}-1$;

$$\mathcal{G}\left(y_{d,g}^f u_a^c\right) = \mathfrak{o} + \mathfrak{s} - 4\alpha(\mathcal{N}-1) - 2\beta(g-1) - (f-d+1) - \beta. \tag{46}$$

Case 15: For $1 \leq a \leq \mathcal{R}$;

$$\mathcal{G}(v_a^c) = \alpha\mathcal{N} + 2\beta\mathcal{R} - (a-1) - \mathcal{R}(c-1). \tag{47}$$

Case 16: For $1 \leq g \leq \mathcal{R}-1,\ d = 1, 2$;

$$\mathcal{G}\left(z_{d,g}^f\right) = 2\alpha\mathcal{N} + 2\beta\mathcal{R} + f + \beta(d-1) + 2\beta(g-1) + 2\ \beta(\mathcal{R}-1) + 4\alpha(\mathcal{N}-1). \tag{48}$$

Case 17: For $d = 1$, with $1 \leq a \leq \mathcal{R}-1$,$1 \leq g \leq \mathcal{R}-1$;

$$\mathcal{G}\left(v_a^c h_{d,g}^f\right) = \mathfrak{o} + \alpha\mathcal{N} + \beta\mathcal{R} + (\alpha+\beta)(a-1) + f + \alpha, \tag{49}$$

$$\mathcal{G}\left(z_{d,g}^f z_{d+1,g}^f\right) = \mathfrak{o} + \mathfrak{s} - 4\alpha(\mathcal{N}-1) - 2\alpha(g-1) - (f-d) - 2\beta(\mathcal{R}-1). \tag{50}$$

Case 18: For $d = 2$, with $2 \leq a \leq \mathcal{R}-1, g+1 = a,$ and, $1 \leq g \leq \mathcal{R}-1$;

$$\mathcal{G}\left(z_{d,g}^f v_a^c\right) = \mathfrak{o} + \mathfrak{s} - 4\alpha(\mathcal{N}-1) - 2\beta(g-1) - 2\beta(\mathcal{R}-1) - (f-d+1) - \beta. \tag{51}$$

Case 19: For $1 \leq a \leq \mathcal{R}-1$;

$$\mathcal{G}(u_a^c v_a^c) = \mathfrak{o} + \alpha\mathcal{N} + \beta\mathcal{R} - \alpha - (a-1)(\alpha+\beta) - (c-1). \tag{52}$$

Case 20: For $a = \mathcal{R}$;

$$\mathcal{G}(u_a^c v_a^c) = \mathfrak{o} + \beta - (c-1). \tag{53}$$

It is simple to acquit that for every subcycle $C_8^\jmath : 1 \leq \jmath \leq \alpha(\mathcal{N}-1) + \beta(\mathcal{R}-1)$ of $G \cong \alpha\mathcal{LP}_n \cup \beta\mathcal{LP}_\mathcal{R}$the weight of $C_8^\jmath$ is $93(\alpha\mathcal{N} + \beta\mathcal{R}) - 58(\alpha+\beta) + 8$. Hence$G \cong \alpha\mathcal{LP}_n \cup \beta\mathcal{LP}_\mathcal{R}$ isC_8-supermagic. □

In the following theorem, we consider C_8-supermagic labeling for disjoint union of non-isomorphic copies of $\mathcal{N}$-polyomino Zig-Zag chain.

Theorem 0.4. *For some positive integers β, α and $\mathcal{N} \geq 3$, the graph $G \cong \beta\mathcal{ZP}_{\mathcal{N}-1} \cup \alpha\mathcal{ZP}_\mathcal{R}$, where $\mathcal{R} = \mathcal{N} - i$, and $1 \leq i \leq \mathcal{N}-2$ is C_8-supermagic.*

Proof. Consider $\mathfrak{o} = |V(G)|$ and $\mathfrak{s} = |E(G)|$. Then $\mathfrak{o} = 2\beta\left(\mathcal{N} + 2\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + 2\left\lceil\frac{\mathcal{N}}{2} - 1\right\rceil\right) + 2\alpha\left(\mathcal{R} + 2\left\lfloor\frac{\mathcal{R}}{2}\right\rfloor + 2\left\lceil\frac{\mathcal{R}}{2} - 1\right\rceil\right)$, $\mathfrak{s} = \beta\left(3\mathcal{N} - 2 + 4\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + 4\left\lceil\frac{\mathcal{N}}{2} - 1\right\rceil\right) + \alpha\left(3\mathcal{R} - 2 + 4\left\lfloor\frac{\mathcal{R}}{2}\right\rfloor + 4\left\lceil\frac{\mathcal{R}}{2} - 1\right\rceil\right)$.

For our convenience the vertex and edge sets of graph G as follows:

$$
\begin{aligned}
\mathbf{V}(\mathbf{G}) = &\left\{s_a^b,\ t_a^b : 1 \leq a \leq \mathcal{N}, 1 \leq b \leq \beta\right\} \cup \left\{u_a^h : 1 \leq a \leq \mathcal{R}, 1 \leq h \leq \alpha\right\} \cup \left\{v_a^h : 1 \leq a \leq \mathcal{R}, 1 \leq h \leq \alpha\right\} \\
&\cup \left\{w_{d,g}^c : 1 \leq c \leq \beta, 1 \leq g \leq \mathcal{N}-1, 1 \leq d \leq 4\right\} \cup \left\{x_{d,g}^c : 1 \leq c \leq \beta, 2 \leq g \leq \mathcal{N}-1, 1 \leq d \leq 4\right\} \\
&\cup \left\{y_{d,f}^e : 1 \leq e \leq \alpha, 1 \leq f \leq \mathcal{R}-1, 1 \leq d \leq 4\right\} \cup \left\{z_{d,f}^e : 1 \leq e \leq \alpha, 2 \leq f \leq \mathcal{R}-1, 1 \leq d \leq 4\right\}, \\
\mathbf{E}(\mathbf{G}) = &\left\{s_a^b t_a^b : 1 \leq a \leq \mathcal{R}, 1 \leq b \leq \beta\right\} \cup \left\{u_a^h v_a^h : 1 \leq a \leq \mathcal{R}-1, 1 \leq h \leq \alpha\right\} \\
&\cup \left\{s_a^b t_a^b : \mathcal{R}+1 \leq a \leq \mathcal{N}, 1 \leq b \leq \beta\right\} \cup \left\{u_a^h v_a^h : a = \mathcal{R}, 1 \leq h \leq \alpha\right\} \\
&\cup \left\{s_a^b w_{d,g}^c : 1 \leq c = b \leq \beta, d = 1, 1 \leq g \leq \mathcal{R}-1, 1 \leq a \leq \mathcal{R}-1\right\} \\
&\cup \left\{s_a^b w_{d,g}^c : 1 \leq c = b \leq \beta, d = 1, \mathcal{R} \leq a \leq \mathcal{N}-1, \mathcal{R} \leq g \leq \mathcal{N}-1\right\} \\
&\cup \left\{t_a^b t_{a+1}^b : 1 \leq b \leq \beta, 1 \leq a \leq \mathcal{R}\right\} \cup \left\{s_a^b s_{a+1}^b : 1 \leq b \leq \beta, 2 \leq a \leq \mathcal{R}-1\right\} \\
&\cup \left\{s_a^b s_{a+1}^b : 1 \leq b \leq \beta, \mathcal{R} \leq a \leq \mathcal{N}-1\right\} \cup \left\{t_a^b x_{d,g}^c : 1 \leq b = c \leq \beta, d = 1, 1 \leq a \leq \mathcal{R}, 1 \leq g \leq \mathcal{R}\right\} \\
&\cup \left\{t_a^b x_{d,g}^c : 1 \leq b = c \leq \beta, d = 1, \mathcal{R}+1 \leq a \leq \mathcal{N}-1, \mathcal{R}+1 \leq g \leq \mathcal{N}-1\right\} \\
&\cup \left\{t_a^b t_{a+1}^b : 1 \leq b \leq \beta, \mathcal{R}+1 \leq a \leq \mathcal{N}-1\right\} \cup \left\{u_a^h u_{a+1}^h : 1 \leq h \leq \alpha, 2 \leq a \leq \mathcal{R}-2\right\} \\
&\cup \left\{u_a^h u_{a+1}^h : 1 \leq h \leq \alpha, a = \mathcal{R}-1\right\} \cup \{u_a^h y_{d,f}^e : 1 \leq h \leq \alpha, 1 \leq e \leq \alpha, a = \mathcal{R}-1, d = 1, \\
&f = \mathcal{R}-1\} \cup \left\{v_a^h v_{a+1}^h : 1 \leq h \leq \alpha, 1 \leq a \leq \mathcal{R}-1\right\} \\
&\cup \left\{v_a^h z_{d,f}^e : 1 \leq e \leq \alpha, 1 \leq h \leq \alpha, d = 1, 2 \leq a \leq \mathcal{R}-1, 2 \leq f \leq \mathcal{R}-1\right\} \\
&\cup \left\{w_{d,g}^c w_{d+1,g}^c : 1 \leq d \leq 3, 1 \leq c \leq \beta, 1 \leq g \leq \mathcal{N}-1\right\} \\
&\cup \left\{x_{d,g}^c x_{d+1,g}^c : 1 \leq d \leq 3, 1 \leq c \leq \beta, 2 \leq g \leq \mathcal{N}-1\right\} \\
&\cup \left\{y_{d,f}^e y_{d+1,f}^e : 1 \leq d \leq 3, 1 \leq e \leq \alpha, 1 \leq f \leq \mathcal{R}-1\right\} \\
&\cup \left\{z_{d,f}^e z_{d+1,f}^e : 1 \leq d \leq 3, 1 \leq e \leq \alpha, 1 \leq f \leq \mathcal{R}-1\right\} \\
&\cup \left\{w_{d,g}^c s_a^b : d = 4, 1 \leq g \leq \mathcal{N}-1, 2 \leq a \leq \mathcal{N}, 1 \leq b \leq \beta\right\} \\
&\cup \left\{x_{d,g}^c t_a^b : d = 4, 1 \leq g \leq \mathcal{N}-1, 3 \leq a \leq \mathcal{N}, 1 \leq b \leq \beta, 1 \leq c \leq \beta\right\} \\
&\cup \left\{y_{d,f}^e u_a^h : d = 4, 1 \leq g \leq \mathcal{R}-1, 2 \leq a \leq \mathcal{R}, 1 \leq h \leq \alpha, 1 \leq e \leq \alpha\right\} \\
&\cup \left\{z_{d,f}^e v_a^h : d = 4, 2 \leq f \leq \mathcal{R}-1, 3 \leq a \leq \mathcal{R}, 1 \leq e \leq \alpha, 1 \leq h \leq \alpha\right\} \\
&\cup \left\{u_a^h y_{d,f}^e : d = 1, 1 \leq f \leq \mathcal{R}-2, 1 \leq a \leq \mathcal{R}-2, 1 \leq e \leq \alpha, 1 \leq h \leq \alpha\right\}. \qquad (54)
\end{aligned}
$$

Define a total labeling $\mathcal{G} : V(G) \cup E(G) \to \left\{1, 2, \ldots, \beta\left(5\mathcal{N} - 2 + 8\left\lfloor \frac{\mathcal{N}}{2} \right\rfloor + 8\left\lceil \frac{\mathcal{N}}{2} - 1 \right\rceil\right) + \alpha\left(5\mathcal{R} - 2 + 8\left\lfloor \frac{\mathcal{R}}{2} \right\rfloor + 8\left\lceil \frac{\mathcal{R}}{2} - 1 \right\rceil\right)\right\}$

Throughout the following labeling, we will consider $1 \leq b \leq \beta,\ 1 \leq h \leq \alpha,\ 1 \leq c \leq \beta,\ 1 \leq e \leq \alpha$.

Case 1: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}\left(s_a^b\right) = a + \mathcal{N}(b-1).$$

Case 2: For $1 \leq d \leq 4$, with $1 \leq g \leq \mathcal{N}-1$, $g = \text{odd}$;

$$\mathcal{G}\left(w_{d,g}^c\right) = 2\beta\mathcal{N} + 2\alpha\mathcal{R} + 2\beta(g-1) + (c-1) + \beta(d-1) + 1.$$

Case 3: For $1 \leq d \leq 3$, with $1 \leq g \leq \mathcal{N}-1$, $g = \text{odd}$;

$$\mathcal{G}\left(w_{d,g}^c w_{d+1,g}^c\right) = \mathfrak{o} + \mathfrak{s} - \beta(d-1) - 2\beta(g-1) - (c-1).$$

Case 4: For $d = 1$;
with $1 \leq g \leq \mathcal{R}-1, 1 \leq a \leq \mathcal{R}-1$, $a = \text{odd}$, $g = \text{odd}$;

$$\mathcal{G}\left(s_a^b w_{d,g}^c\right) = \mathfrak{o} + \beta(2\mathcal{N}-1) + \alpha(2\mathcal{R}-1) + d + (\beta+\alpha)(a-1) + (b-1),$$

with $\mathcal{R} \leq g \leq \mathcal{N}-1, \mathcal{R} \leq a \leq \mathcal{N}-1$, $a=$ odd, $g=$ odd;

$$\mathcal{G}\left(s_a^b w_{d,g}^c\right) = \mathfrak{o} + \beta\left(2\mathcal{N}-1\right) + \alpha\left(2\mathcal{R}-1\right) + d + \left(\beta+\alpha\right)\left(\mathcal{R}-2\right) + \left(b-1\right) + \beta\left(1+a-\mathcal{R}\right).$$

Case 5: For $d=4$, with $1 \leq g \leq \mathcal{N}-1$, $2 \leq a \leq \mathcal{N}$, $g=$ odd,$a=$ even;

$$\mathcal{G}\left(w_{d,g}^c s_a^b\right) = \mathfrak{o} + \mathfrak{s} - \beta\left(d-1\right) - \left(b-1\right) - 2\beta\left(g-1\right).$$

Case 6: For $2 \leq a \leq \mathcal{R}-1$, $a=$ even;

$$\mathcal{G}\left(s_a^b s_{a+1}^b\right) = \mathfrak{o} + \beta\left(2\mathcal{N}-1\right) + \alpha\left(2\mathcal{R}-1\right) + \left(a-2\right)\left(\beta+\alpha\right) + \left(b-1\right) + 1 + \left(\beta+\alpha\right).$$

Case 7: For $1 \leq a \leq \mathcal{R}$;

$$\mathcal{G}\left(s_a^b p_a^b\right) = \mathfrak{o} + \beta\mathcal{N} + \alpha\mathcal{R} - \left(\beta+\alpha\right)\left(a-1\right) - \left(b-1\right).$$

Case 8: For $\mathcal{R}+1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}\left(s_a^b t_a^b\right) = \mathfrak{o} + \beta\left(\mathcal{N}-\mathcal{R}\right) + \alpha - \beta\left(a-\left(\mathcal{R}+1\right)\right) - \left(b-1\right).$$

Case 9: For $\mathcal{R} \leq a \leq \mathcal{N}-1$, $a=$ even;

$$\mathcal{G}\left(s_a^b s_{a+1}^b\right) = \mathfrak{o} + \beta\left(2\mathcal{N}-1\right) + \alpha\left(2\mathcal{R}-1\right) + \left(\mathcal{R}-2\right)\left(\beta+\alpha\right) + \beta\left(1+a-\mathcal{R}\right) + \left(b-1\right) + 1.$$

Case 10: For $1 \leq a \leq \mathcal{N}$;

$$\mathcal{G}\left(t_a^b\right) = 2\beta\mathcal{N} + 2\alpha\mathcal{R} - \mathcal{N}\left(b-1\right) - a + 1.$$

Case 11: For $1 \leq d \leq 4$, $2 \leq g \leq \mathcal{N}-1$, $g=$ even;

$$\mathcal{G}\left(x_{d,g}^c\right) = 2\beta\mathcal{N} + 2\alpha\mathcal{R} + 2\beta\left(g-2\right) + \left(c-1\right) + 4\beta\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + 1 + \beta\left(d-1\right).$$

Case 12: For $1 \leq d \leq 3$, $2 \leq g \leq \mathcal{N}-1$, $g=$ even;

$$\mathcal{G}\left(x_{d,g}^c x_{d+1,g}^c\right) = \mathfrak{o} + \mathfrak{s} - \beta\left(d-1\right) - 2\beta\left(g-2\right) - \left(c-1\right) - 4\beta\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor.$$

Case 13: For $d=1$
with $2 \leq a \leq \mathcal{R}, 2 \leq g \leq \mathcal{R}$, $a=$ even, $g=$ even;

$$\mathcal{G}\left(t_a^b x_{d,g}^c\right) = \mathfrak{o} + \beta\mathcal{N} + \alpha\mathcal{R} + \left(a-1\right)\left(\beta+\alpha\right) + \left(b-1\right) + d + \left(\beta+\alpha\right).$$

with $\mathcal{R}+1 \leq a \leq \mathcal{N}-1, \mathcal{R}+1 \leq g \leq \mathcal{N}-1$, $a=$ even, $g=$ even;

$$\mathcal{G}\left(t_a^b x_{d,g}^c\right) = \mathfrak{o} + \beta\mathcal{N} + \alpha\mathcal{R} + \mathcal{R}\left(\beta+\alpha\right) + \beta\left(a-\left(\mathcal{R}+d\right)\right) + \left(b-d\right) - \left(\alpha-d\right).$$

Case 14: For $d=4$,$2 \leq g \leq \mathcal{N}-1$, $3 \leq a \leq \mathcal{N}$, $a=$ odd, $g=$ even;

$$\mathcal{G}\left(x_{d,g}^c t_a^b\right) = \mathfrak{o} + \mathfrak{s} - \beta\left(4\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + \left(d-1\right) + 2\left(g-2\right)\right) - \left(b-1\right).$$

Case 15: For $1 \leq a \leq \mathcal{R}$, $a=$ odd;

$$\mathcal{G}\left(t_a^b t_{a+1}^b\right) = \mathfrak{o} + \beta\mathcal{N} + \alpha\mathcal{R} + \left(a-1\right)\left(\beta+\alpha\right) + \left(b-1\right) + 1.$$

Case 16: For $\mathcal{R}+1 \leq a \leq \mathcal{N}-1$, $a =$ odd;

$$\mathcal{G}\left(t_a^b t_{a+1}^b\right) = \mathfrak{o} + \beta\mathcal{N} + \alpha\mathcal{R} + \mathcal{R}\left(\beta+\alpha\right) + \beta\left(a-\left(\mathcal{R}+1\right)\right) + \left(b-1\right) - \left(\alpha-1\right).$$

Case 17: For $1 \leq a \leq \mathcal{R}$;

$$\mathcal{G}\left(u_a^h\right) = \beta\mathcal{N} + a + \mathcal{R}\left(h-1\right).$$

Case 18: For $1 \leq d \leq 4$, $1 \leq f \leq \mathcal{R}-1$, $f =$ odd;

$$\mathcal{G}\left(y_{d,f}^e\right) = 2\left(\beta\mathcal{N} + \alpha\mathcal{R} + 2\beta\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + 2\beta\left\lceil\frac{\mathcal{N}}{2}-1\right\rceil + \alpha\left(f-1\right)\right) + 1 + \alpha\left(d-1\right) + \left(e-1\right).$$

Case 19: For $1 \leq d \leq 3$, $1 \leq f \leq \mathcal{R}-1$, $f =$ odd;

$$\mathcal{G}\left(y_{d,f}^e y_{d+1,f}^e\right) = \mathfrak{o} + \mathfrak{s} - \alpha\left(d-1\right) - 2\alpha\left(f-1\right) - \left(e-1\right) - 4\beta\left(\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + \left\lceil\frac{\mathcal{N}}{2}-1\right\rceil\right).$$

Case 20: For $d = 1$;
with $a = \mathcal{R}-1, f = \mathcal{R}-1$, $a =$ odd, $f =$ odd;

$$\mathcal{G}\left(u_a^h y_{d,f}^e\right) = \mathfrak{o} + \beta\left(2\mathcal{N}-1\right) + \alpha\left(2\mathcal{R}-1\right) + \beta\left(\mathcal{N}-d\right) + \alpha\left(\mathcal{R}-2\right) + d + \left(a-\mathcal{R}+d\right) + \left(h-d\right),$$

with $1 \leq a \leq \mathcal{R}-2, 1 \leq f \leq \mathcal{R}-2$;

$$\mathcal{G}\left(u_a^h y_{d,f}^e\right) = \mathfrak{o} + \beta\left(2\mathcal{N}-1\right) + \alpha\left(2\mathcal{R}-1\right) + d + \beta + \left(\beta+\alpha\right)\left(a-1\right) + \left(h-1\right).$$

Case 21: For $d = 4$, $1 \leq f \leq \mathcal{R}-1$,$2 \leq a \leq \mathcal{R}$, $f =$ odd, $a =$ even;

$$\mathcal{G}\left(y_{d,f}^e u_a^h\right) = \mathfrak{o} + \mathfrak{s} - 4\beta\left(\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + \left\lceil\frac{\mathcal{N}}{2}-1\right\rceil\right) - \alpha\left(d-1\right) - 2\alpha\left(f-1\right) - \left(h-1\right).$$

Case 22: For $2 \leq a \leq \mathcal{R}-2$, $a =$ even;

$$\mathcal{G}\left(u_a^h u_{a+1}^h\right) = \mathfrak{o} + \beta\left(2\mathcal{N}-1\right) + \alpha\left(2\mathcal{R}-1\right) + \beta + 1 + \left(\beta+\alpha\right)\left(a-1\right) + \left(h-1\right).$$

Case 23: For $a = \mathcal{R}-1$, $a =$ even;

$$\mathcal{G}\left(u_a^h u_{a+1}^h\right) = \mathfrak{o} + \beta\left(2\mathcal{N}-1\right) + \alpha\left(2\mathcal{R}-1\right) + \beta\left(\mathcal{N}-1\right) + \alpha\left(\mathcal{R}-2\right) + \left(a-\mathcal{R}+2\right) - \left(h-1\right).$$

Case 24: For $1 \leq a \leq \mathcal{R}$;

$$\mathcal{G}\left(v_a^h\right) = \alpha\mathcal{N} + 2\beta\mathcal{R} - \left(a-1\right) - \mathcal{R}\left(h-1\right).$$

Case 25: For $1 \leq d \leq 4$, $2 \leq f \leq \mathcal{R}-1$, $f =$ even;

$$\mathcal{G}\left(z_{d,f}^e\right) = 2\left(\beta\mathcal{N} + \alpha\mathcal{R} + 2\beta\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + 2\beta\left(\left\lceil\frac{\mathcal{N}}{2}-1\right\rceil\right) + \alpha\left(f-2\right) + 2\alpha\left\lfloor\frac{\mathcal{R}}{2}\right\rfloor\right) + \alpha\left(d-1\right) + 1 + \left(e-1\right).$$

Case 26: For $1 \leq d \leq 3$, $2 \leq f \leq \mathcal{R}-1$, $f =$ even;

$$\mathcal{G}\left(z_{d,f}^e z_{d+1,f}^e\right) = \mathfrak{o} + \mathfrak{s} - \alpha\left(d-1\right) - \left(e-1\right) - 4\beta\left(\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + \left\lceil\frac{\mathcal{N}}{2}-1\right\rceil\right) - 2\alpha\left(2\left\lfloor\frac{\mathcal{R}}{2}\right\rfloor + \left(f-2\right)\right).$$

Case 27: For $d = 1$, $2 \le a \le \mathcal{R} - 1$,$2 \le f \le \mathcal{R} - 1$, $a = $ even, $f = $ even;

$$\mathcal{G}\left(v_a^h z_{d,f}^e\right) = \mathfrak{o} + \beta\left(\mathcal{N} + d\right) + \alpha\mathcal{R} + \left(\beta + \alpha\right)\left(a - 2\right) + 2\left(\beta + \alpha\right) + \left(h - 1\right) - \beta.$$

Case 28: For $d = 4$, $2 \le f \le \mathcal{R} - 1$,$3 \le a \le \mathcal{R}$, $a = $ odd, $f = $ even;

$$\mathcal{G}\left(z_{d,f}^e v_a^h\right) = \mathfrak{o} + \mathfrak{s} - 4\beta\left(\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + \left\lceil\frac{\mathcal{N}}{2} - 1\right\rceil\right) - \alpha\left(\left(d - 1\right) + 2\left(f - 2\right) + 4\left\lfloor\frac{\mathcal{R}}{2}\right\rfloor\right) - \left(h - 1\right).$$

Case 29: For $1 \le a \le \mathcal{R} - 1$, $a = $ odd;

$$\mathcal{G}\left(v_a^h v_{a+1}^h\right) = \mathfrak{o} + \beta\mathcal{N} + \alpha\mathcal{R} + \beta + 1 + \left(\beta + \alpha\right)\left(a - 1\right) - \left(h - 1\right).$$

Case 30: For $1 \le a \le \mathcal{R} - 1$;

$$\mathcal{G}\left(u_a^h v_a^h\right) = \mathfrak{o} + \beta\mathcal{N} + \alpha\mathcal{R} - \left(\beta + \alpha\right)\left(a - 1\right) - \left(h - 1\right) - \beta.$$

Case 31: For $a = \mathcal{R}$;

$$\mathcal{G}\left(u_a^h v_a^h\right) = \mathfrak{o} + \alpha - \left(h - 1\right).$$

To show that $\mathcal{G}$ is a C_8-supermagic labeling of G, imagine C_8^n, $1 \le n \le \beta\left(\mathcal{N} - 1\right) + \alpha\left(\mathcal{R} - 1\right)$, be the subcycles of $G \cong \beta\mathcal{ZP}_{\mathcal{N}-1} \cup \alpha\mathcal{ZP}_{\mathcal{R}}$. Since the weight of every cycle C_8^n is

$$45\left(\beta\mathcal{N} + \alpha\mathcal{R}\right) + 48\beta\left\lfloor\frac{\mathcal{N}}{2}\right\rfloor + 48\beta\left\lceil\frac{\mathcal{N}}{2} - 1\right\rceil + 48\alpha\left\lfloor\frac{\mathcal{R}}{2}\right\rfloor + 48\alpha\left\lceil\frac{\mathcal{R}}{2} - 1\right\rceil - 10\left(\beta - 1\right) - 10\left(\alpha - 1\right) - 12$$

therefore, G is a C_8-supermagic. □

Conclusion

When a graph having cycle sub-graphs either the graph itself is single component or disjoint union of non-isomorphic copies of original graphs, by looking into a graph in the shape of cycle sub-graphs and labeling each vertex and edge such that each cycle is obtained by same magic and also while assigning numbering priorities given to the vertices is called as cycle-super magic labeling. In this paper, we study some polyomino structures, including zig-zag and linear chains. We studied, C_8-super magic labeling of zig-zag, linear chains and also the disjoint union of non-isomorphic copies of both chains.

Data Availability

The datasets used and/or analysed during the current study available from the corresponding author on reasonable request.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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