SCIENTIFIC OASIS

Journal of Operations Intelligence

Journal homepage: www.jopi-journal.org
ISSN: 3009-4267

Connecting the Numerical Scale Model With Assessing Attitudes and its Application to Hesitant Fuzzy Linguistic Multi-attribute Decision Making

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ARTICLE INFO

Article history:
Received 12 September 2024
Received in revised form 2 October 2024
Accepted 11 November 2024
Available online 14 November 2024

Keywords:
Multiple attribute decision making, Numerical scale, Hesitant fuzzy linguistic term set, Assessing attitudes, Weights.

ABSTRACT

The aim of this paper is to provide a specific numerical scale model with the purpose of making transformations between linguistic terms and numerical values. The proposed method represents a wide range of existing numerical scale models, and can quantitatively reflect the linguistic behaviors of DMs. The pessimistic-optimistic principle based supplementary regulation for hesitant fuzzy linguistic term set (HFLTS) may lead to the initial information distortion and losing. On the basis of the lowest common multiple principle in number theory, an improved supplementary regulation is proposed to reserve the fidelity of original information, the improved supplementary regulation brings a new conception for information measures of HFLTS as well. Then based on the traditional generalized distance and Hausdorff distance measures, some new distance measures for HFLTS are presented in the numerical scale framework. Furthermore, an extended TOPSIS method for hesitant fuzzy linguistic MADM is developed. Finally, a numerical example concerning the preference of movies is elaborated on the performance of our approach. Sensitive and comparative analysis are also provided and discussed to show the effectiveness and advantages of the proposed method

1. Introduction

Along with the development of the world economy, decision making has become closely linked with our daily routine. Multiple attribute decision making (MADM), which tends to select the most appropriate alternative(s) by a decision maker (DM) who expresses his/her preferences to various attributes, has been regarded as one of the most significant activities in various fields [1–3].

A crucial issue in the decision making process is the way DMs express their preferences. Those

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https://doi.org/10.31181/jopi31202531

chosen for expressing DMs' information are of high importance because only reasonable approaches can effectively and fully express information of DMs. Considering it is hard for DMs to access information by quantitative values, the linguistic approach [4–6], initially proposed by Zadeh, turned out to be an effective tool for handing practical problems. In recent years, different linguistic models [7–9] have been proposed for Computing With Words (CWW). For instance, the 2-tuple linguistic representation models [10, 11], the virtual linguistic term set (VLTS) [12] and the numerical scale (NS) models [13–15]. Among them, particularly the NS models, which are based on an one to one mapping between the linguistic information and numerical values (NVs), are now the most direct and effective ones which have been successfully applied to various linguistic decision making problems. However, all these models assess evaluation only by a single linguistic term (LT), such a single LT is often insufficient. Motivated by the hesitant fuzzy sets (HFSs) [16] and the linguistic term sets (LTSs), the hesitant fuzzy linguistic terms set (HFLTS), introduced by Rodríguez et al. [17], makes the representation of linguistic information more flexible and efficient by using several LTs simultaneously. Some further and latest researches on HFLTS can be found in [18–24].

In the process of expression of views, it is not hard to find the evaluation results are influenced not only by the experience and subjective awareness of DMs but also by their assessing attitudes. Without any doubt, people have different cognitive styles such as thinking, sensing, feeling and intuitive styles [25]. All these influence each other and affect the mental models of DMs. Consequently, these cognitive styles have a direct influence on how DMs form their beliefs (how they decide the 'goodness' or 'poorness') and use such beliefs to make judgments. Specifically, there are two main patterns:

- The use of LTs: When facing the same evaluation model, DMs may use different forms of LTs models to express their judgments. If they are sure of their knowledge and skills, they may give a crisp LT, i.e., "Very Good"; If they can not make a determined judgement, they may give their answers by interval LTs, i.e., between "Good" and "Very Good"; and some provide a more complex term, for instance, DMs are willing to provide a set of possible LTs, which also known as the HFLTS. Apparently, forcing DMs to give a determined judgment is not necessary. Thus, the HFLTS is an effective tool and it provides many advantages in depicting DMs' cognition and preferences.

- Two DMs may both say that the research value is "Good", but one is linguistic-conservative, while the other is linguistic-radical. In such case, the NV of linguistic-conservative assessment may be 0.8, while the NV of linguistic-radical assessment may be 0.6 (Here the NV is in [0,1]), thus they should not be treated equally [25, 26]. Similarly, even if two DMs express different LTs like "Good" and "Very good", respectively, their NVs can be the same considering their different assessing attitudes. Besides, positive judgments (LTs related to "Good") should not be treated equally with the negative judgments (LTs related to "Bad").

To better reflect individual personalized differences in understanding LTs, Li et al. [27, 28] have proposed several personalized individual semantics (PIS) approaches by means of NSs and 2-tuple linguistic model. These approaches show good features for managing linguistic information in CWW processes. In fact, individual linguistic evaluation is closely related to their assessing attitudes. From the point of utility theory, the same LTs may mean different utilities to different people. The process we discuss here reflects LTs and their characteristics is shown in Fig. 1. Taking each individual assessing attitudes into consideration, the motivations of this paper are threefold:

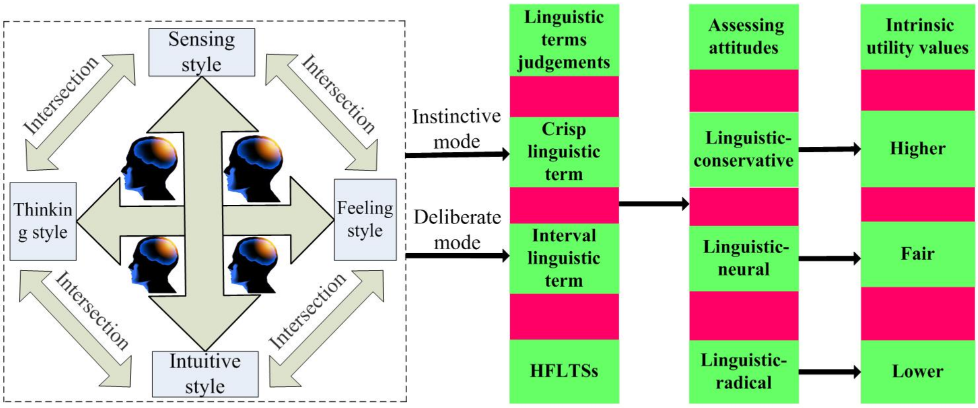

Figure 1: The numerical values and assessing attitudes based on different cognitive styles

- Despite the desirable properties of existing linguistic models in establishing relations between LTs and NVs, few of them takes the DM's assessing attitudes into consideration at a deeper level. The PIS approaches proposed by Li et al. [26–28] mostly focused on consistency-driven optimization-based models to obtain the PIS. Although Wang et al. [29] gave several linguistic scale functions (LSFs), they have not specified how to choose LSFs in real applications, and these LSFs may only be used to reflect some specific assessing attitudes of DMs. Thus, we connect the NS model and provide a specific NS representation model considering the assessing attitudes of DMs. Our proposed NS model has a reasonably simple mathematical expression and can represent a wide range of NS models, it can commendably reflect the equivalent transformation between LTs and NVs as well. Moreover, estimating the parameters of the NS from NVs will fit the model compatibly and satisfactorily.

- Most of the existing normalization approaches of HFLTS are based on the risk preferences of DMs, the evident drawback is that they may lead to the loss of original information by adding some artificial elements, by this way, the pessimistic-optimistic based normalization principle is no longer useful and available here (For details please see **Example 6**). These realizations motive us to provide an improved normalization algorithm for HFLTS to achieve the highest level of retention of information and develop a series of new distance measures of HFLTSs.

- In practical MADM problems, the weights of attributes should be derived firstly if they are unknown. From an objective perspective, if the deviation degree of an attribute is smaller than others, it should be assigned a smaller weight, otherwise, it should be assigned a larger weight [30]. Considering the difficulty in measuring the deviation among LTs, it is of theoretical and practical significance to solve it by using the NVs obtained by the NS model. Thus, we extend the new developed distance measures of HFLTSs to reflect the deviation among attributes, and derive the weights of attributes with different assessing attitudes.

To achieve above contents, the rest of this paper is allocated as follows. In Section 2, three linguistic representation models are reviewed briefly. Section 3 connects the NS model with assessing attitudes and proposes a specific NS representation model, this model shows good features of PIS in understanding the meaning of LTs. The HFLTS representation model, improved comparison law, novel supplementary regulation for HFLTS are introduced in Section 4, some new distance measures are provided as well. Section 5 proposes an approach to MADM based on the extended TOPSIS method in which a model for optimal weighting vector is constructed. In Section 6, a numerical example is provided to illustrate the efficiency of the proposed method, sensitive and comparative analysis are also discussed in this section. Section 7 gives some concluding remarks.

2. Preliminaries

Before DMs provide their preferences over an object with linguistic labels, a proper linguistic evaluation scale is needed. In this section, we briefly blue review three linguistic representation models proposed by Xu [31], Herrera and Martínez [10, 11], Dong et al. [13–15], respectively.

Let $S = \{s_t | t = 0, 1, ..., g\}$ be a linguistic term set (LTS) with odd cardinality, the LT s_t denotes the $t + 1$th LT of S and represents a possible value for a linguistic variable, $g + 1$ is a non-negative real number and denotes the granularity of S. S must have the following characteristics [32]:
(1) Ordered: $s_i \geq s_j$, if $i \geq j$;
(2) A negation operator: $neg(s_i) = s_j$, such that $j = g - i$.

An example of LTS S with seven LTs is given as

$$\begin{aligned} S = \{&s_0 = VB(\text{Very bad}), s_1 = B(\text{Bad}), s_2 = SB(\text{Somewhat Bad}), s_3 = F(\text{Fair}), \\ &s_4 = SG(\text{Somewhat good}), s_5 = G(\text{Good}), s_6 = VG(\text{Very good})\}. \end{aligned} \tag{1}$$

Herrera and Martínez [10, 11] contributed a 2-tuple linguistic representation model. This model defines a function making transformations between linguistic 2-tuples and NVs.

Definition 2.1 ([10, 11]) Let $\beta \in [0, g]$ be a number in the granularity interval of the LTS $S = \{s_t | t = 0, 1, ..., g\}$, $t = round(\beta)$ and $\alpha = \beta - t$ be two values such that $t \in [0, g]$ and $\alpha \in [-0.5, 0.5)$. Then, α is called a symbolic translation, with $round$ being the usual rounding operation.

Definition 2.2 ([10, 11]) Let $S = \{s_t | t = 0, 1, ..., g\}$ be a LTS, $\beta \in [0, g]$ is a value of the symbolic result, then the 2-tuple that expresses the equivalent information to β is obtained with the following functions:

$$\Delta : [0, g] \to S \times [-0.5, 0.5), \tag{2}$$

$$\Delta(\beta) = (s_t, \alpha), \text{ with } \begin{cases} s_t, & t = round(\beta), \\ \alpha = \beta - t, & \alpha \in [-0.5, 0.5). \end{cases} \tag{3}$$

Clearly, Δ is one to one, and Δ has an inverse function $\Delta^{-1} : \bar{S} \to [0, g]$ with $\Delta^{-1}((s_t, \alpha_t)) = t + \alpha_t$.

The 2-tuple linguistic model has some computational properties.
(1) A 2-tuple comparison operator: $(s_i, \alpha_i) \geq (s_j, \alpha_j)$, if and only if $\Delta^{-1}(s_i, \alpha_i) \geq \Delta^{-1}(s_j, \alpha_j)$;
(2) A 2-tuple negation operator: $neg((s_i, \alpha_i)) = \Delta(g - \Delta^{-1}(s_i, \alpha_i))$;
(3) Some 2-tuple linguistic aggregation operators [6,7].

Xu [31] extended S to a continuous form $\bar{S} = \{s_\alpha | \alpha \in [0, g]\}$. Due to the lack of syntax and semantics, Xu [31] stated that the VLTs can only appear in operations. Xu and Wang [12] generated a new VLT by a symbolic transformation as follows:

Definition 2.3 ([12]) Let $S = \{s_t | t = 0, 1, ..., g\}$ be a LTS with the semantics defined on the domain U. For any $t = 0, 1, ..., g$, let

$$\delta \in \begin{cases} [0, 0.5), & t = 0 \\ [-0.5, 0], & t = g \\ [-0.5, 0.5), & else \end{cases} \tag{4}$$

then the pair (t, δ) generates a VLT s_α, with $\alpha = t + \delta$. The set of VLTs is denoted by $\bar{S} = \{s_\alpha | \alpha \in [0, g]\}$.

Dong et al. [33] pointed out that the key in the calculation of linguistic representation models is to define a function transforming linguistic 2-tuples into NVs. They further extended the 2-tuple fuzzy linguistic representation models by formally proposing the concept of numerical scale.

Definition 2.4 ([33]) Let $S = \{s_t | t = 0, 1, ..., g\}$ be a LTS and R be real number set, the mapping $NS : S \to R$ is a numerical scale of S and $NS(s_t)$ is called the numerical index of s_t.

Definition 2.5 ([33]) For $(s_t, \alpha_t) \in \bar{S}$, the numerical scale $\overline{NS}$ on $\bar{S}$ is defined by

$$\overline{NS}\left((s_t, \alpha_t)\right) = \begin{cases} NS(s_t) + \alpha_t \times (NS(s_{t+1}) - NS(s_t)), & \alpha_t \geq 0 \\ NS(s_t) + \alpha_t \times (NS(s_t) - NS(s_{t-1})), & \alpha_t < 0. \end{cases} \tag{5}$$

By selecting different NSs, different distributions of LTS S can be obtained (see Figs. 2-4):

1. The symmetric and uniform LTSs. In this circumstance, the absolute deviations between two adjacent LTs are equal and the negation operator is always correct;

2. The symmetric but non-uniform LTSs. In this circumstance, the absolute deviations between two adjacent LTs are not equal, but the negation operator is always correct;

3. The asymmetric and non-uniform LTS. In this situation, the absolute deviations between two adjacent LTs may increase, decrease or remain unchanged, and the negation operator may not involve one-to-one mapping.

It is worth mentioning that the asymmetric and non-uniform LTS is more consistent with the actual distribution of LTs and more flexible to reflect the utilities of DMs with assessing attitudes than the first and the second LTSs.

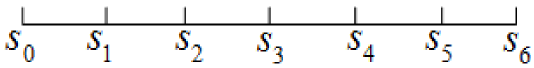

Figure 2: A set of seven symmetric and uniform LTs

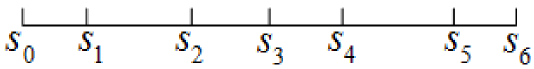

Figure 3: A set of seven symmetric and non-uniform LTs

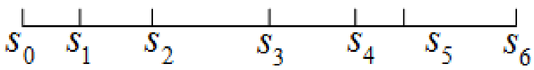

Figure 4: A set of seven asymmetric and non-uniform LTs

3. Connecting the numerical scale model with assessing attitudes

3.1 S-Shaped and Non-S-Shaped utility functions

Utility function (UF) is an important concept to measure preferences when modelling uncertainty. UFs include S-shaped and non-S-shaped UFs. Each UF $u(x)$ affords a framework including the basic function form, loss aversion coefficient and reference point. Usually, u is a non-decreasing function, that is if $u(x') > u(x)$, then the DM prefers x' to x.

Risk aversion is the behavior of humans when they are exposed to uncertainty. In fact, the higher the curvature of $u(x)$, the higher the risk aversion. One important measure is the Arrow-Pratt measure, named with the economists Arrow and Pratt [34].

$$R_A = -u''(x)/u'(x). \tag{6}$$

It can quantitatively measure a DM's risk attitudes, and the DM is said to be:
(1) **Risk-averse**: if $R_A = -u''(x)/u'(x) > 0$;
(2) **Risk-neutral**: if $R_A = -u''(x)/u'(x) = 0$;
(3) **Risk-seeking**: if $R_A = -u''(x)/u'(x) < 0$.

The most famous theory characterizing DMs' psychological factors is the prospect theory's value function presented by Tversky and Kahneman [35], such that

$$v(x) = \begin{cases} x^{\alpha} & x \geq 0 \\ -\lambda(-x)^{\beta} & x < 0 \end{cases} \tag{7}$$

where $\alpha, \beta \in (0, 1]$ and $\lambda > 1$.

Shalev [36] developed a non-S-shaped UF. The feature of non-S-shaped UF is that the utility curve for relative losses or gains is flexible, meaning that non-S-shaped UF regards a wide range of assessing attitudes.

3.2 A specific numerical scale representation model

3.2.1 Existing numerical scales

The NS itself is a mathematical representation that quantifies the LTs by their corresponding NVs. The definition of NS is put forward as:

Definition 3.1 Let $S = \{s_t | t = 0, 1, ..., g\}$ be a linguistic term set, the NS is a mapping:

$$NS : s_t \to \gamma, \gamma \in [0, 1], \tag{8}$$

the symbol $\gamma \in [0, 1]$ indicates the numerical value of the LT s_t. The mapping NS satisfies the following properties:
(1) **Boundedness**: $NS(s_t) \in [0, 1]$;
(2) **Minimum element**: $NS(s_0) \in [0, 0.5)$;
(3) **Neutral element**: $NS(s_{g/2}) = 0.5$;
(4) **Maximum element**: $NS(s_g) \in (0.5, 1]$;
(5) **Monotone increasing**: $NS(s_{t_1}) \geq NS(s_{t_2})$ if $t_1 \geq t_2$.

Remark 3.1 Property (1) ensures the numerical values of LTs fall into the interval $[0, 1]$, which bears an uncanny resemblance to the membership function.

Remark 3.2 Property (2) ensures the numerical value of the lower bound of the LTS falls into the interval $[0, 0.5)$, which strengthens the condition $NS(s_0) = 0$ in [17,30], because when the granularity of the LTS S is not that big such that $S = \{s_0(\text{Very bad}), s_1(\text{bad}), s_2(\text{medium}), s_3(\text{good}), s_4(\text{Very good})\}$, it is more reasonable to set $NS(s_0) \in [0, 0.5)$.

Remark 3.3 Similar to the property (2), property (4) strengthens the condition $NS(s_g) = 1$ in [17,30], it is more reasonable to set $NS(s_g) \in (0.5, 1]$.

Remark 3.4 Property (3) ensures the numerical value of the medium LT $s_{g/2}$ means there is almost no difference between any two objects.

Remark 3.5 Property (5) reflects the NS is an increasingly ordered numerical scale.

Wang et al. [29] provided several LSFs. Bao et al. [37] set up an exponential scale model. Zhou and Xu [38] proposed a general asymmetric linguistic term set (GALTS), and so on. These NSs are flexible and can give deterministic results according to different semantics. They are shown as:
(1) A symmetric and uniform computing model

$$NS_1(s_t) = t/g, \tag{9}$$

where the NVs of LTs are divided on average.
(2) A symmetric and non-uniform computing model [29]

$$NS_2(s_t) = \begin{cases} \frac{a^{g/2}-a^{g/2-t}}{2a^{g/2}-2}, & (t=0,1,...,g/2) \\ \frac{a^{g/2}+a^{t-g/2}-2}{2a^{g/2}-2}, & (t=g/2+1,...,g), \end{cases} \tag{10}$$

where a can be determined using a subjective method.
(3) An asymmetric and non-uniform computing model [38, 39]

$$NS_3(s_t) = \begin{cases} \left(1+e^{-\delta_1 t}\right)^{-1}, & t \geq 0 \\ \left(1+e^{-\delta_2 t}\right)^{-1}, & t < 0, \end{cases} \tag{11}$$

$NS_3(\cdot)$ is based on the additive LTS and sigmoid function, $\delta_1, \delta_2 \geq 0$ are two risk appetites parameters for "good" and "poor" of the semantics, respectively.
(4) An asymmetric and non-uniform computing model [29]

$$NS_4(s_t) = \begin{cases} \frac{(g/2)^\alpha-(g/2-t)^\alpha}{2(g/2)^\alpha}, & (t=0,1,...,g/2) \\ \frac{(g/2)^\beta+(t-g/2)^\beta}{2(g/2)^\beta}, & (t=g/2+1,...,g), \end{cases} \tag{12}$$

where $\alpha, \beta \in (0,1]$, $NS_4(\cdot)$ has the similar form as the value function (7) in prospect theory.

For more details about them, please see [29, 38–40]. The above NS models can nicely reflect some desired properties such as the asymmetry and non-uniformity of LTSs. Some handling methods and characteristics of them are shown as:

• The NS and PIS models [26–28].

1. They established consistency-driven optimization models to compute the NVs of LTs, and they paid more attention to the objective consistency of the linguistic preference relation (LPR).
2. Their consistency-driven methodology focused on linguistic GDM resolution framework, and the ranking of alternatives was derived from the preference relations.

• The LSFs [29].

1. The LSFs in [29] are preferable in practice, they are flexible and can give deterministic results according to different semantics.
2. The operations were defined based on the LSFs, and some interval-valued hesitant fuzzy linguistic aggregation operators were proposed further.

• The GALTS based on the sigmoid semantics [38, 39].

1. They constructed a generalized and asymmetric LTS and applied it to the qualitative decision making involving risk appetites.
2. They designed an value-at-risk fitting (VARF) approach for obtaining the risk appetite parameters of the GALTS.

3.2.2 Our specified numerical scale model for linguistic terms

It is not hard to find that the convexity of the NS mainly depends on the DMs' cognitive styles. Historically, there has been few research conducted to empirically identify relevant functional forms with respect to the NS. Thus, it is of theoretical and practical significance to establish a specific NS representation model with flexible concavity. In general, the core idea of our numerical scale model includes two parts

1. Establish a NS model that can satisfactorily approximate the relations between LTs and NVs;

2. Establish a NS model that can quantitatively reflect the assessing attitudes of PIS.

To achieve the above two points, our method of dealing with them is given as

1. To approximate the relations between LTs and NVs, some LTs and their corresponding NVs are subjectively provided by each individual, it is also a natural premise which can be achieved easily.

2. To satisfactorily approximate the relations between LTs and NVs, the polynomial curve fitting which can acquire good fitting effect is considered here.

3. To quantitatively reflect the assessing attitudes of PIS, some parameters which can be aggregated into a measure index is necessary.

Without loss of generality, our NS representation model consists of a basic UF (polynomial function), loss aversion coefficient 'orness' measure and a reference point $s_{g/2}$. They are explained as follow:
(1) Basic function form: Fractional polynomial function.

Fractional polynomial curve fitting is an usually used data fitting method. When there is not too many data for fitting, polynomial curve function can acquire good fitting effect.
(2) Assessing attitudes: 'orness' measure

The 'orness' measure was presented by Yager [42], which is also called the attitudinal character of the aggregation. Here, we use it to reflect the assessing attitudes of DMs.
(3) A reference point: $s_{g/2}$

The middle LT $s_{g/2}$ is chosen as a reference point, because it means there is almost no difference between any two objects.

Any polynomial function has the form as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \tag{13}$$

Observe Eq. (12), consider $i = g/2 + 1, ..., g$, we have

$$NS_4(s_i) = \frac{(g/2)^\beta + (i - g/2)^\beta}{2(g/2)^\beta} = \frac{1}{2} + \frac{1}{2}((i - g/2)(2/g))^\beta$$

Let $(i - g/2)(2/g) = \Delta t$, then it becomes

$$NS_4(\Delta t) = \frac{1}{2} + \frac{1}{2}(\Delta t)^\beta,$$

which is just a power function similar to Eq. (7). Furthermore, any power function is a special type of polynomial function.

A NS model which can commendably reflect the concavity and convexity based on the fractional polynomial function is constructed as:

$$NS(\Delta t) = \begin{cases} \frac{1}{2} - \frac{1}{2}\sum\limits_{i=1}^{n} w_i^1 \Delta t^{\alpha_i}, & (i = 0, 1, ..., g/2) \\ \frac{1}{2} + \frac{1}{2}\sum\limits_{i=1}^{n} w_i^2 \Delta t^{\beta_i}, & (i = g/2+1, ..., g) \end{cases} \tag{14}$$

which satisfies $\sum\limits_{i=1}^{n} w_i^1 \leq 1, \sum\limits_{i=1}^{n} w_i^2 \leq 1, w_i^1, w_i^2 \in [0,1], \alpha_i, \beta_i \in (0, +\infty)$ and $\alpha_i \leq \alpha_{i+1}, \beta_i \leq \beta_{i+1}$.

The NS model (14) is constituted of the linear combination of series of power functions, therefore, it has desirable properties and preferable approach ability. Further, when the number of LTs is not too big, a fractional polynomial function with less terms is enough. Consider a specific NS representation model with less terms as:

$$NS(\Delta t) = \frac{1}{2} \pm \frac{1}{2}(\underbrace{w_1\Delta t^{\beta_1} + w_2\Delta t^{\beta_2}}_{Risk\ averse\ part} + \underbrace{w_3\Delta t}_{Risk\ neural\ part} + \underbrace{w_4\Delta t^{\beta_4} + w_5\Delta t^{\beta_5}}_{Risk\ seeking\ part})\ , \tag{15}$$

which satisfies $\sum\limits_{i=1}^{5} w_i \leq 1, w_i \in [0,1]$, $\Delta t = (i - g/2)(2/g)$ and $0 < \beta_1 \leq \beta_2 < 1 < \beta_4 \leq \beta_5$.

The specific NS model (15) is constructed on the basis of the linear combination of three types of power functions, that is, the risk-averse part, the risk-neural part and the risk-seeking park. The power function $f(x) = x^\beta$ shows good linkage to the concavity and convexity. The weighted coefficient in the fractional polynomial can be used to quantitatively describe the assessing attitudes.

Proposition 3.1 When setting $w_1 = 1$ or $w_2 = 1$, we have $NS(\Delta t) = \frac{1}{2} + \frac{1}{2}\Delta t^\beta$, which turns into the NS model (12).

Proposition 3.2 When setting $w_3 = 1$, we have $NS(\Delta t) = \frac{1}{2} + \frac{1}{2}\Delta t$, which turns into the NS model (9).

Proposition 3.3 The setting $\sum\limits_{i=1}^{n} w_i^1 \leq 1$ and $\sum\limits_{i=1}^{n} w_i^2 \leq 1$ lead to $NS(s_0) \in (0, 0.5)$ and $NS(s_g) \in (0.5, 1)$, which are consist with the properties (2) and (4) in Definition 3.1.

Proposition 3.4 By means of the Taylor expansion technique, the exponential function $f(t) = e^{-\delta_1 t}$ can be expanded in Taylor's series, thus our NS model can successfully approximate NSs (10) and (11).

Proposition 3.5 Our proposed NS model is symmetric if and only if $w_i^1 = w_i^2, \alpha_i = \beta_i$.

Proposition 3.6 Our proposed NS model (15) can be viewed as a specific OWA computing model as it must satisfy $\Delta t^{\beta_1} \geq \Delta t^{\beta_2} > \Delta t > \Delta t^{\beta_4} \geq \Delta t^{\beta_5}$ with $0 < \beta_1 \leq \beta_2 < 1 < \beta_4 \leq \beta_5$ and $\Delta t \in [0,1]$.

Proposition 3.7 When fixed β_i are set to approximate our NS model, the 'orness' measure $\mathrm{orness}(w) = \alpha = \frac{1}{n-1}\sum\limits_{i=1}^{n}(n-i)w_i$ can be used quantitatively to describe the assessing attitudes.

Our proposed NS model can satisfactorily denote the concavity and convexity of the subjective NSs for inferior LTs and superior LTs respectively. For instance, if the DM is linguistic-conservative, then the larger weights are assigned to the risk averse part, and if the DM is linguistic-radical, then the larger weights are assigned to the risk seeking part. Furthermore, by setting fixed β_i in model (15), if the DM is linguistic-conservative, then the larger 'orness' measure is obtained, conversely, then the smaller 'orness' measure is obtained.

As a new version of linguistic representation model, our proposed NS model can nicely reflect some desirable properties like asymmetry, non-uniformity and so on [38]. Some specific numerical scale models are shown in Fig. 5.

As is well known, different DMs have different assessing attitudes toward LTs. Thus an important

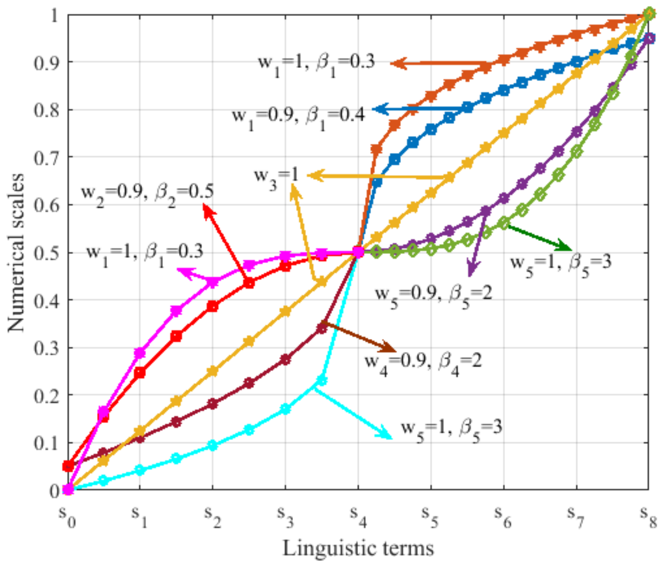

Figure 5: Numerical scale model for linguistic terms with different assessing attitudes

issue using the new NS is to obtain the suitable parameters that can represent a DM's assessing attitudes towards LTs.

To address it, an optimal discrete fitting (ODF) technology is designed as follows:

Let $S = \{s_t | t \in [0, g]\}$ be a set of LTs, $NS(s_t)$ be its NS, the DM's subjective accepted NVs is $V = \{v(s_o), ..., v(s_p), ...v(s_q)\}, o, p, q \in [0, g]$. The ODF approach to approximate our NS model includes the following three steps:

Step 1: Choose the fixed β_i to approximate our NS model.

Tversky and Kahneman [35] suggested $\alpha = 0.88$ (Eq. (7)) according to their experimental results, thus we set $\beta_2 = 0.88$ in Eq. (15). Besides, a power function with moderate strong up-convex form where $\beta_1 = 0.3$ is selected. Correspondingly, two power functions are nearly symmetric with them are set such that $\beta_4 = 1.5$ and $\beta_5 = 2$ here. Obviously, others fixed β_i can be also predetermined in our NS model, the difference is there exists different fitting precisions when setting different fixed β_i.

Step 2: Construct the optimal constraint. Here we consider $i > g/2$, the other part can be obtained the same way, where

$$\min_{NS} \sum_{t>g/2} |NS(s_t) - v(s_t)| \\ s.t. \begin{cases} \sum_{i=1}^{5} w_i \leq 1 \\ w_i \in [0, 1] \end{cases} \tag{16}$$

Step 3: Solve the above programming problem and obtain the optimal fitting weights w_i. Then we can obtain the orness measure and the corresponding assessing attitudes of DMs.

Next, the following example is provided to demonstrate our ODF approach.

Example 1. Let S be a LTS as Eq. (1), suppose a DM provides his/her NVs $V = \{v(s_0) = 0, v(s_1) = 0.2, v(s_2) = 0.3, v(s_4) = 0.7, v(s_5) = 0.8, v(s_6) = 1\}$. Based on the ODF technology, the optimal constraint is built as

$$\min_{NS} \sum_{t>3} |NS(s_t) - v(s_t)| \quad \text{and} \quad \min_{NS} \sum_{t\leq 3} |NS(s_t) - v(s_t)| \\ s.t. \begin{cases} \sum_{i=1}^{5} w_i^2 \leq 1 \\ w_i^2 \in [0, 1] \end{cases} \qquad s.t. \begin{cases} \sum_{i=1}^{5} w_i^1 \leq 1 \\ w_i^2 \in [0, 1] \end{cases}$$

Solve the above model and we obtain $w_1^2 = 0.4685, w_2^2 = 0.0066, w_3^2 = 0.0065, w_4^2 = 0.0097, w_5^2 = 0.5087$, therefore, the orness measure $\alpha_2 = 0.5697$. Similarly, the orness measure $\alpha_1 = 0.4791$, these two results lead to the DM is weakly inferior linguistic radical and weakly superior linguistic conservative. The corresponding NSs are shown in Fig. 6.

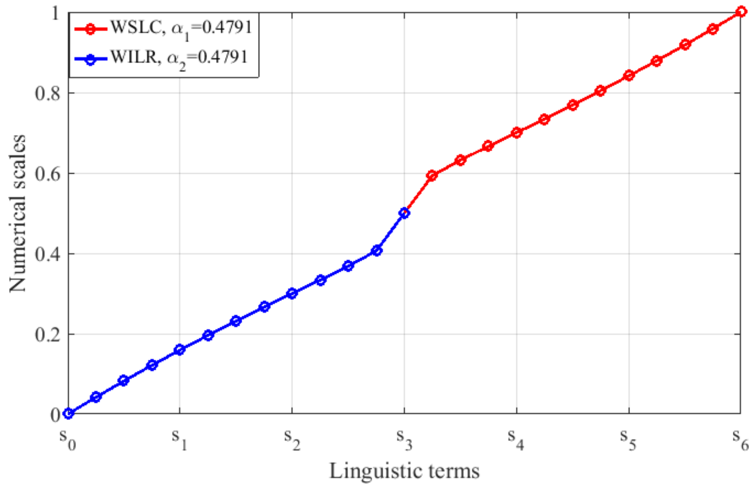

Figure 6: The NSs with $\alpha_1 = 0.4791$ and $\alpha_2 = 0.5697$

3.2.3 Methodological contributions compared with the existed linguistic representation models

In this subsection, we point out some advantages and methodological contributions compared with the existed linguistic representation models.

• Compared with the NS and PIS models [26–28].

1. Li et al. [26–28] obtained the PIS by the LPR. If DMs express their opinions via other preference structures, their proposal can not be employed. While our method is more simple, and our model has advantage of low computational complexity;

2. In their consistency-driven methodology, only the LPRs reach acceptable consistency can the NVs of LTs be obtained. After the adjustment of the original LPR, the change of the elements in the original LPR is unavoidable, this may lead to the loss of information.

• Compared with the NSs in [29, 37, 40].

1. Although different types of LSFs are proposed, they can only be used to reflect some specific assessing attitudes of DMs, thus how to choose LSFs is still questionable. Our proposed NS model generalizes a wide range of existing NS models.

2. Unlike the LSFs which use different functional forms describe the assessing attitudes, our model firstly obtains the optimal parameters by the ODF technology, and then utilizes the ‘orness’ measure to quantitatively describe the assessing attitudes.

• Compared with the GALTS in [38, 39].

1. The asymmetric sigmoid semantics satisfy $g(t) = 1$ and $g(t) = 0$ with considerable parameters δ_1 and δ_2, which make it hard in determining the functional form in real cases.

2. The Arrow-Pratt coefficient of absolute risk aversion of $g(t)$ satisfies $R_g \geq 0$, therefore, this semantics model can be only used to describe the numerical values of LTs who have a marginal decreasing effect.

4. Hesitant fuzzy linguistic term sets representation model

4.1 *The improved comparison law for HFLEs*

Rodríguez et al. [17] defined the concept of HFLTS:

Definition 4.1 ([17]) Let $S = \{s_0, s_1, ..., s_g\}$ be a LTS. A hesitant fuzzy linguistic term set (HFLTS) H_s, is an ordered finite subset of the consecutive linguistic terms of S.

Definition 4.2 ([17]) For three HFLTSs H_S, H_S^1 and H_S^2, the following operations are defined:
(1) Lower bound: $H_S^- = \min(s_t) = s_k, s_t \in h_S$.
(2) Upper bound: $H_S^+ = \max(s_t) = s_k, s_t \in h_S$.
(3) $H_S^c = \{s_t | s_t \in S \ and \ s_t \notin H_S\}$.
(4) $H_S^1 \cup H_S^2 = \{s_t | s_t \in H_S^1 \ or \ s_t \in H_S^2\}$.
(5) $H_S^1 \cap H_S^2 = \{s_t | s_t \in H_S^1 \ and \ s_t \in H_S^2\}$.
(6) Envelope of a HFLTS: $env(H_S) = [H_S^-, H_S^+]$.

Rodríguez et al. [17] initially used the comparison theory of interval values to rank HFLTSs based on the envelop of the HFLTS. Their approach can satisfactorily make comparison among HFLTSs for most cases, but there may exist a few special cases fail in making comparison. The following example is an indicator, where,

Example 2. Let $h_{S_1} = \{s_1, s_2, s_3, s_4, s_5\}$, $h_{S_2} = \{s_2, s_3, s_4\}$ and $h_{S_3} = \{s_3\}$ be three HFLEs. Let $\mathrm{Ind}(h_S)$ be the set of indexes of the LTs in a HFLTS h_S on S, then we get

$$p(h_{S_1} > h_{S_2}) = p(h_{S_1} > h_{S_3}) = p(h_{S_2} > h_{S_3}) = 0.5.$$

By Rodríguez's approach, we have $h_{S_1} = h_{S_2} = h_{S_3}$. However, it is obvious $h_{S_1} \neq h_{S_2} \neq h_{S_3}$, therefore, using the ranking method presented in [17] to compare HFLEs, they are not totally ordered.

Liao et al. [41] defined the mean and the hesitant degree of a HFE, inspired by them, we can define the mean and the dispersion degree of a HFLE in a more general form:

Definition 4.4 Let $h_S = \{s_{\theta_l} | s_{\theta_l} \in S, l = 1, 2, ..., \#h_S\}$ be a HFLE, $s_{\theta_l} \in S$ being the possible LTs of h_S, and $\#h_S$ being the length of h_S. The mean of HFLE h_S is defined as:

$$\bar{h}_S = NS^{-1}(\frac{1}{\#h_S}\sum\nolimits_{l=1}^{\#h_S} NS(s_{\theta_l})). \tag{17}$$

Definition 4.5 Let h_S be a HFLE. The dispersion degree of HFLE h_S is defined as:

$$D(h_S) = \sqrt{(\sum\nolimits_{l=1}^{\#h_S} (NS(s_{\theta_l}) - NS(\bar{h}_S))^2)/\#h_S}. \tag{18}$$

Example 3. Following example 2, let $NS(s_t) = t/g$, then $\bar{h}_{S_1} = \bar{h}_{S_2} = \bar{h}_{S_3} = s_3$, and $D(h_{S_1}) = 0.2357, D(h_{S_2}) = 0.1361, D(h_{S_3}) = 0$, which means that the dispersion degree of h_{S_1} is greater than that of h_{S_2} and h_{S_3}.

He et al. [42] defined the ith-order polymerization degree functions to compare different HFSs. From the statistical point of view, the bigger the value of $p_i(h)$, the more stable of the values in HFE, then the bigger of the HFE. Next, we would like to define the ith-order polymerization degree of a HFLE in a more general form.

Definition 4.5 For a HFLE h_S,

$$p_i(h_S) = 1 - (\sum\nolimits_{l=1}^{\#h_S} |NS(s_{\theta_l}) - NS(\bar{h}_S)|^i/\#h_S)^{1/i}, \tag{19}$$

is called the ith-order polymerization degree of h_S, where $\bar{h}_S$ is mean of the HFLE h_S defined by Eq. (17).

Based on the mean and the polymerization degree functions of HFLEs, we define the new comparison law for different HFLEs as follows.

Definition 4.6 For two HFLEs h_{S_1} and h_{S_2}, then we have
(1). If $\bar{h}_{S_1} > \bar{h}_{S_2}$, then $h_{S_1} > h_{S_2}$;
(2). If $\bar{h}_{S_1} < \bar{h}_{S_2}$, then $h_{S_1} < h_{S_2}$;
(3). If $\bar{h}_{S_1} = \bar{h}_{S_2}$, then there exists $u, v \in \{2, ..., n\}$, with the condition that $u < v$
1). If $p_u(h_{S_1}) > p_u(h_{S_2})$, then $h_{S_1} > h_{S_2}$;
2). If $p_u(h_{S_1}) < p_u(h_{S_2})$, then $h_{S_1} < h_{S_2}$;
3). If $p_u(h_{S_1}) = p_u(h_{S_2})$, then
(i). If $p_v(h_{S_1}) > p_v(h_{S_2})$, then $h_{S_1} > h_{S_2}$;
(ii). If $p_v(h_{S_1}) < p_v(h_{S_2})$, then $h_{S_1} < h_{S_2}$.

Example 4. Following example 2. The three HFLEs can't be ranked by the method in [17], if we use the polymerization degree function and let $NS(s_t) = t/g$, we have $\bar{h}_{S_1} = \bar{h}_{S_2} = \bar{h}_{S_3} = s_3$, $p_1(h_{S_1}) = 0.8, p_1(h_{S_2}) = 0.8889, p_1(h_{S_3}) = 1$. Thus $h_{S_3} > h_{S_2} > h_{S_1}$, which is consistent with our thought.

4.2 *Improved supplementary regulation for HFLEs based on the lowest common multiple principle*

The most representative supplementary regulation for HFLEs is the pessimistic-optimistic principle proposed by Xu and Xia [43], Liao et. al [44], what they have in common is that the shorter one is extended by adding any value between a min and a max until they have the same length.

It seems reasonable to take the DMs' preferences into account, but it may lead to the initial information distortion and losing. Three important measures in Section IV for HFLEs can directly reflect the possible LTs. Therefore, one of our significant principles of supplementary regulation for HFLEs is to remain them unchanged. Stimulated by [45, 46], we firstly give the concept of r-HFLE as follows:

Definition 4.7 Let h_S be a HFLE with $\#h_S$ LTs, then h_S^r is a multiset with $r \cdot \#h_S$ elements defined by

$$h_S^r = \{\underbrace{s_{\theta_1}, ..., s_{\theta_1}}_{r\ times}, \underbrace{s_{\theta_2}, ..., s_{\theta_2}}_{r\ times}, ..., \underbrace{s_{\theta_{\#h_S}}, ..., s_{\theta_{\#h_S}}}_{r\ times}\}, \tag{20}$$

the multiplicity of each element s_{θ_l} of h_S in h_S^r is r, and we call h_S^r the r-times hesitant fuzzy linguistic element (r-HFLE).

The idea for supplementing n HFLEs $h_{S_i} (i = 1, 2, ..., n)$ of different lengths is repeating their elements. Considering the lowest common multiple (lcm) of $\#h_{S_i} (i = 1, 2, ..., n)$, denoted by $L = lcm(\#h_{S_1}, \#h_{S_2}, ..., \#h_{S_n})$, we can repeat every element of h_{S_i} $L/\#h_{S_i}$ times, then $h_{S_i} (i = 1, 2, ..., n)$ are extended into $L/\#h_{S_i}$-times HFLEs with the form of

$$h_{S_1}^{L/\#h_{S_1}} = \{\underbrace{s_{\theta_1}^{h_{S_1}}, ..., s_{\theta_1}^{h_{S_1}}}_{L/\#h_{S_1}\ times}, \underbrace{s_{\theta_2}^{h_{S_1}}, ..., s_{\theta_2}^{h_{S_1}}}_{L/\#h_{S_1}\ times}, ..., \underbrace{s_{\theta_{\#h_{S_1}}}^{h_{S_1}}, ..., s_{\theta_{\#h_{S_1}}}^{h_{S_1}}}_{L/\#h_{S_1}\ times}\}$$

$$h_{S_2}^{L/\#h_{S_2}} = \{\underbrace{s_{\theta_1}^{h_{S_2}}, ..., s_{\theta_1}^{h_{S_2}}}_{L/\#h_{S_2}\ times}, \underbrace{s_{\theta_2}^{h_{S_2}}, ..., s_{\theta_2}^{h_{S_2}}}_{L/\#h_{S_2}\ times}, ..., \underbrace{s_{\theta_{\#h_{S_2}}}^{h_{S_2}}, ..., s_{\theta_{\#h_{S_2}}}^{h_{S_2}}}_{L/\#h_{S_2}\ times}\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h_{S_n}^{L/\#h_{S_n}} = \{\underbrace{s_{\theta_1}^{h_{S_n}}, ..., s_{\theta_1}^{h_{S_n}}}_{L/\#h_{S_n}\ times}, \underbrace{s_{\theta_2}^{h_{S_n}}, ..., s_{\theta_2}^{h_{S_n}}}_{L/\#h_{S_n}\ times}, ..., \underbrace{s_{\theta_{\#h_{S_n}}}^{h_{S_n}}, ..., s_{\theta_{\#h_{S_n}}}^{h_{S_n}}}_{L/\#h_{S_n}\ times}\}$$

Example 5. Let us consider $A = \{s_0, s_1\}$ and $B = \{s_0, s_1, s_2\}$. The lcm of 2 and 3 is 6, then $A^3 = \{s_0, s_0, s_0, s_1, s_1, s_1\}$ and $B^2 = \{s_0, s_0, s_1, s_1, s_2, s_2\}$.

Theorem 4.1 Let h_S be a HFLE, and $\bar{h}_S$ be the mean of h_S. Assume that h_S^r is the r-HFLE of h_S, then the mean of h_S^r is still $\bar{h}_S$.

Proof: By Eqs. (17) and (20), the mean of r-HFLE $h_{\tilde{S}}^r$ is

$$\begin{aligned}\bar{h}_S^r &= NS^{-1}(\textstyle\sum_{k=1}^{r\#h_S} NS(s_{\theta_k})/r\#h_S)\\ &= NS^{-1}(\textstyle\sum_{k=1}^{\#h_S} rNS(s_{\theta_k})/r\#h_S)\\ &= NS^{-1}(\textstyle\sum_{k=1}^{\#h_S} NS(s_{\theta_k})/\#h_S) = \bar{h}_S,\end{aligned}$$

which completes the proof of Theorem 4.1.

Theorem 4.2 Let $D(h_S)$ be the dispersion of h_S, then the dispersion degree of its r-HFLE h_S^r is equal to $D(h_S)$.

Proof: By Eq. (18), the dispersion degree of r-HFLE h_S^r is

$$\begin{aligned}D(h_S^r) &= \sqrt{\textstyle\sum_{k=1}^{r\#h_S} \left(NS(s_{\theta_k}) - NS(\bar{h}_S^r)\right)^2/r\#h_S}\\ &= \sqrt{\textstyle\sum_{k=1}^{r\#h_S} \left(NS(s_{\theta_k}) - NS(\bar{h}_S)\right)^2/r\#h_S}\\ &= \sqrt{\textstyle\sum_{k=1}^{\#h_S} r\left(NS(s_{\theta_k}) - NS(\bar{h}_S)\right)^2/r\#h_S}\\ &= \sqrt{\textstyle\sum_{k=1}^{\#h_S} \left(NS(s_{\theta_k}) - NS(\bar{h}_S)\right)^2/\#h_S}\\ &= D(h_S),\end{aligned}$$

which completes the proof of Theorem 4.2.

Theorem 4.3 Let $p_i(h_S)$ be the ith-order polymerization degree of h_S, then the ith-order polymerization degree of h_S^r:

$$p_i(h_S^r) = p_i(h_S). \tag{21}$$

Proof: According to Eq. (19),

$$\begin{aligned}&p_i(h_S^r)\\ &= 1 - (\textstyle\sum_{k=1}^{\#h_S r} |NS(s_{\theta_l}) - \sum_{k=1}^{\#h_S r} NS(s_{\theta_k})/\#h_S r|^i/\#h_{\tilde{S}} r)^{1/i}\\ &= 1 - (\textstyle\sum_{k=1}^{\#h_S r} |NS(s_{\theta_l}) - \sum_{k=1}^{\#h_S} NS(s_{\theta_k})/\#h_S|^i/\#h_S r)^{1/i}\\ &= 1 - (\textstyle\sum_{k=1}^{\#h_S r} |NS(s_{\theta_l}) - NS(\bar{h}_S)|^i/\#h_S r)^{1/i}\\ &= 1 - (\textstyle\sum_{k=1}^{\#h_S} r|NS(s_{\theta_l}) - NS(\bar{h}_S)|^i/\#h_S r)^{1/i}\\ &= 1 - (\textstyle\sum_{k=1}^{\#h_S} |NS(s_{\theta_l}) - NS(\bar{h}_S)|^i/\#h_S)^{1/i}\\ &= p_i(h_S),\end{aligned}$$

which completes the proof of Theorem 4.3.

Theorems 4.1-4.3 show that, by using the improved supplementary regulation for HFLEs on the basis of the LCM principle, some important measures of HFLEs remain unchanged. Therefore, it is more reasonable to use this method in dealing with HFLEs that are of different lengths.

4.3 *New distance measures for HFLTSs via improved supplementary regulation*

Recently, Liao et al. [44] set out investigation of distance measures for HFLTSs.

Definition 4.8 ([44]) Let h_{S_1} and h_{S_2} be two HFLEs, then the distance measure between h_{S_1} and h_{S_2} is defined as $d(h_{S_1}, h_{S_2})$, which satisfies:

(1) $0 \le d(h_{S_1}, h_{S_2}) \le 1$;

(2) $d(h_{S_1}, h_{S_2}) = 0$ if and only if $h_{S_1} = h_{S_2}$;

(3) $d(h_{S_1}, h_{S_2}) = d(h_{S_2}, h_{S_1})$.

Liao et al. [44] introduced the normalized generalized distance for HFLEs as follows:

$$d_{GD}(h_{S_1}, h_{S_2}) = ((\sum_{l=1}^{L} |\theta_l^1 - \theta_l^2|/(2\tau+1))^\lambda/L)^{1/\lambda}. \tag{22}$$

As mentioned before, the pessimistic-optimistic principle [21,46] may lead to the initial information distortion, and then lead to unidentifiable results. The following example can be used to explain this limitation.

Example 6. Let S be a LTS and $NS(s_t) = t/g$ be its NS. Assume there exist two patterns represented by HFLEs $h_{S_1} = \{s_{2.7}, s_{3.6}, s_{4.2}\}$ and $h_{S_2} = \{s_3, s_{3.9}\}$. Consider a sample to be recognized represented by a HFLE $h_S = \{s_{2.7}, s_{3.3}\}$.

Known by the principle of the minimum distance measure between HFLEs, we can judge which the sample h_S belongs to. As HFLEs h_{S_1} and h_S are of different lengths, according to the pessimistic-optimistic principle, such that

$$s_{\bar{b}} = \xi h_S^- \oplus (1-\xi) h_S^+$$

where $\xi (0 \le \xi \le 1)$ is an optimized parameter.

By Eq. (22), and let $\lambda = 1$, we can get the distance measures between h_S and $h_{S_i} (i = 1, 2)$, and then determine which the sample h_S belongs to, results are shown in Table 1.

Table 1: Distance measures between h_S and $h_{S_i} (i = 1, 2)$

$d(h_S, h_{S_i})$	h_{S_1}	h_{S_2}	The sample h_S belongs to
$h_S^{\xi=0} = \{s_{2.7}, s_{2.7}, s_{3.3}\}$	0.0857	0.0643	h_{S_2}
$h_S^{\xi=0.5} = \{s_{2.7}, s_3, s_{3.3}\}$	0.0714	0.0643	h_{S_2}
$h_S^{\xi=1} = \{s_3, s_{3.3}, s_{3.3}\}$	0.0571	0.0643	h_{S_1}

We find the sample h_S belongs to the pattern h_{S_2} under the pessimistic and neutral principles, while it belongs to the pattern h_{S_1} under the optimistic principle. In fact, the subjects may be machine but not person, there is no physics or emotion in their body, we can't recognize the sample h_S belongs to a fixed pattern. Thus, it is necessary to reconsider the distance measure between HFLEs. Take the improved supplementary regulation into account, some new distance measures between HFLEs are developed as follows.

Definition 4.9 Let h_{S_1} and h_{S_2} be two HFLEs, L be the lcm of $\#h_{S_1}$ and $\#h_{S_2}$, then the normalized generalized distance measure between h_{S_1} and h_{S_2} is defined as

$$d_{GD}(h_{S_1}, h_{S_2}) = (\sum_{l=1}^{L} |NS(s_{\sigma_l}^1) - NS(s_{\sigma_l}^2)|^\lambda/L)^{1/\lambda}, \tag{23}$$

where $\lambda > 0$, $s_{\sigma_l}^1$ and $s_{\sigma_l}^2$ are the lth elements in $h_{S_1}^{L/\#h_{S_1}}$ and $h_{S_2}^{L/\#h_{S_2}}$, respectively.

In particular, if $\lambda = 1$ and $\lambda = 2$, then the above generalized distance becomes the Hamming distance and the Euclidean distance respectively. Similarly, the Hausdorff distance measure can be extended to HFLTS environment. For two HFLEs h_{S_1} and h_{S_2}, the Hausdorff distance measure is defined as:

$$d_{HD}(h_{S_1}, h_{S_2}) = \max_{l=1,2,\dots,L} |NS(s_{\sigma_l}^1) - NS(s_{\sigma_l}^2)|. \tag{24}$$

In addition, we can obtain some hybrid distance measures combining the above distance measures, such as the generalized hybrid distance between h_{S_1} and h_{S_2}:

$$d_{GHD}(h_{S_1}, h_{S_2}) = (\frac{1}{2}(\frac{1}{L}\sum_{l=1}^{L} |NS(s_{\sigma_l}^1) - NS(s_{\sigma_l}^2)|^\lambda + \max_{l=1,2,\dots,L} |NS(s_{\sigma_l}^1) - NS(s_{\sigma_l}^2)|^\lambda))^{1/\lambda}, \lambda > 0. \tag{25}$$

Example 7. Following example 6, three HFLEs can be expressed as $h^2_{S_1} = \{s_{2.7}, s_{2.7}, s_{3.6}, s_{3.6}, s_{4.2}, s_{4.2}\}$, $h^3_{S_2} = \{s_3, s_3, s_3, s_{3.9}, s_{3.9}, s_{3.9}\}$ and $h^3_S = \{s_{2.7}, s_{2.7}, s_{2.7}, s_{3.3}, s_{3.3}, s_{3.3}\}$. By Eq. (23) and let $\lambda = 1$, then we have $d(h_{S_1}, h_S) = 0.0833, d(h_{S_2}, h_S) = 0.075$, thus h_S belongs to the pattern h_{S_2}, which leads to a deterministic result.

In the following, the weighted distance measures for two collections of HFLTSs are defined.

Definition 4.10 For two collections of HFLTSs $H^1_S = \{(h_{S_1})^1, ..., (h_{S_n})^1\}$ and $H^2_S = \{(h_{S_1})^2, ..., (h_{S_n})^2\}$ with associated weighting vector $\omega = (\omega_1, ..., \omega_n)^T$, where $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$, the generalized weighted distance measure between H^1_S and H^2_S is defined as:

$$d_{GWD}(H^1_S, H^2_S) = (\sum\nolimits_{j=1}^n \sum\nolimits_{l=1}^{L_j} \frac{\omega_j}{L_j} |NS(s^{1j}_{\sigma_l}) - NS(s^{2j}_{\sigma_l})|^\lambda)^{1/\lambda}, \tag{26}$$

where $\lambda > 0$, $s^{1j}_{\sigma_l}$ and $s^{2j}_{\sigma_l}$ are the lth elements in $(h_{S_j}^{L/\#h_{S_j}})^1$ and $(h_{S_j}^{L/\#h_{S_j}})^2$, respectively, L_j is the lcm of $\#(h_{S_j})^1$ and $\#(h_{S_j})^2$ for $j = 1, 2, ..., n$, ω_j is the weight of each element $x_j \in X$.

The generalized weighted Hausdorff distance measure between H^1_S and H^2_S is defined as:

$$d_{GWH-D}(H^1_S, H^2_S) = (\sum\nolimits_{j=1}^n \omega_j \max_{l=1,2,...,L_j} |NS(s^{1j}_{\sigma_l}) - NS(s^{2j}_{\sigma_l})|^\lambda)^{1/\lambda}, \lambda > 0. \tag{27}$$

Certainly, we can derive some hybrid weighted distance measures by combining the above distance measures, such as the generalized weighted hybrid distance between H^1_S and H^2_S:

$$\begin{aligned} &d_{GWHD}(H^1_S, H^2_S) = \\ &(\sum\nolimits_{j=1}^n \tfrac{\omega_j}{2} (\sum\nolimits_{l=1}^{L_j} |NS(s^{1j}_{\sigma_l}) - NS(s^{2j}_{\sigma_l})|^\lambda / L_j + \max_{l=1,...,L_j} |NS(s^{1j}_{\sigma_l}) - NS(s^{2j}_{\sigma_l})|^\lambda))^{1/\lambda}, \lambda > 0. \end{aligned} \tag{28}$$

5. MADM problem based on extended TOPSIS method

5.1 Problem description

In this section, we will present the process of solving MADM problem by using the extended TOPSIS method, where the weights of attributes are unknown, the preference values take the form of HFLEs. A MADM problem can be defined as a quadruple $< X, C, A >$, where

$X = \{x_1, x_2, \cdots, x_m\}$ is the discrete set of alternatives for DM and is indexed by i and $m \geq 2$;

$C = \{c_1, c_2, \cdots, c_n\}$ is the set of attributes for each alternative, and the attributes are assumed to be confluent and independent in this paper for simplicity;

$A = (a_{ij})_{m\times n}$ is the decision matrix, and a_{ij} represents the preference value of alternative x_i with respect to attribute c_j, a_{ij} is in the form of HFLEs derived from a given LTS.

An important step in the solution process is the determination of the attributes weights. Before introducing a whole process for handling MADM, we first construct an optimization model to determine the weighting vector of attributes.

5.2 Maximizing deviation method to obtain the attributes weights

Considering the MADM problems where the attribute weights information is completely unknown, based on the maximizing deviation method [30], we extend it to hesitant fuzzy linguistic environment.

Firstly, the DM provides his/her accepted NVs. Based on the ODF technology, the NS model is determined. The generalized deviation degree (GDD) between any two HFLEs is defined as Eq. (23). The GDD between alternative x_i and all the other alternatives with respect to attribute c_j is given as

$$d_{ij}(\omega) = \sum_{i=1}^m \omega_j(d(h_{S_{ij}}, h_{S_{lj}})), i = 1, ..., m, j = 1, ..., n. \tag{29}$$

The idea of maximizing deviation method is that if the GDD among alternatives is small for an attribute, then this attribute should be assigned a small weight, otherwise, it should be assigned a large one. Let

$$d_j(\omega) = \sum_{i=1}^{m} d_{ij}(\omega) = \sum_{l=1}^{m}\sum_{i=1}^{m} \omega_j(d(h_{S_{ij}}, h_{S_{lj}})), \tag{30}$$

denotes the GDD of one alternative and others with respect to the attribute c_j, and then let

$$d(\omega) = \sum_{j=1}^{n} d_j(\omega) = \sum_{j=1}^{n}\sum_{l=1}^{m}\sum_{i=1}^{m} \omega_j(d(h_{S_{ij}}, h_{S_{lj}})), \tag{31}$$

expresses the sum of the GDDs among all attributes.

Then we can construct the following single-objective optimization model to determine the weighting vector ω so as to make the GDD $d(\omega)$ as large as possible, where

$$\begin{aligned} \max \quad & d(\omega) = \sum_{j=1}^{n}\sum_{l=1}^{m}\sum_{i=1}^{m} \omega_j(d(h_{S_{ij}}, h_{S_{lj}})) \\ s.t. \quad & \omega_j \in [0,1], j = 1, ..., n, \sum_{j=1}^{n} \omega_j^2 = 1 \end{aligned} \tag{32}$$

The normalized optimal weighting vector $\omega = (\omega_1, \omega_2, ..., \omega_n)$ can be obtained as follows:

$$\omega_j = \frac{\sum_{l=1}^{m}\sum_{i=1}^{m} d(h_{S_{ij}}, h_{S_{lj}})}{\sum_{j=1}^{n}\sum_{l=1}^{m}\sum_{i=1}^{m} d(h_{S_{ij}}, h_{S_{lj}})}. \tag{33}$$

5.3 *The decision making procedure*

To obtain the best option(s), a new approach based on the extended TOPSIS method is proposed to solve the MADM problem. The new approach involves the following steps:

Step 1. Normalization of the decision matrix.

On one hand, there are two main types of attributes including benefit attributes (J_b) and cost attributes (J_c). To eliminate the effect caused by different types of attributes, the cost attributes can be transformed into the benefit attributes by using the negation operator [47] in HFLTS environment. On the other hand, if some HFLEs in the decision matrix are of different lengths, the improved supplementary regulation for HFLEs is applied to the normalization process. Then we can get the normalized decision matrix.

Step 2. Generation of attribute weights.

Once the normalized decision matrix is obtained, by using the ODF technology, the corresponding NS model is determined. We further utilize the maximizing deviation method to derive the optimal weighting vector ω of attributes.

Step 3. Define the hesitant fuzzy linguistic positive-ideal solution (HFLPIS) and hesitant fuzzy linguistic negative-ideal solution (HFLNIS).

For each HFLTS H_S, according to Definition 4.2, we can obtain the lower bound $H_S^{ij-} = \min_l \{s_{\sigma_l}^{ij}\}$ and the upper bound $H_S^{ij+} = \max_l \{s_{\sigma_l}^{ij}\}$. Thus, we can define the HFLPIS v^+ and the HFLNIS v^- as follows, respectively.

$$V^+ = \{H_S^{1+}, H_S^{2+}, ..., H_S^{n+}\}, \tag{34}$$

$$V^- = \{H_S^{1-}, H_S^{2-}, ..., H_S^{n-}\}, \tag{35}$$

where $H_S^{j+} = \max_{l,j}\{s_{\sigma_l}^{ij}\}$ and $H_S^{j-} = \min_{l,j}\{s_{\sigma_l}^{ij}\}$.

Step 4. Calculate the distance measures between alternative x_i and the HFLPIS and HFLNIS, respectively, the distance measures can be determined by the DM himself/herself, where,

$$d_i^+ = \sum_{j=1}^{n} \omega_j d(a_{ij}, V^+), \tag{36}$$

$$d_i^- = \sum_{j=1}^{n} \omega_j d(a_{ij}, V^-). \tag{37}$$

Step 5. Calculate the relative closeness coefficient (RCC) of alternatives to the HFLPIS as

$$RCC_i = \theta(1 - d_i^+ / \sum_{i=1}^{m} d_i^+) + (1-\theta)(d_i^- / \sum_{i=1}^{m} d_i^-),\ i = 1, ..., m. \tag{38}$$

This new ranking index was proposed by Kuo [48], who treated the separations of an alternative from the PIS and the NIS as a "cost" criterion and a "benefit" criterion, respectively. The parameter θ denotes the importance of the "cost "criterion and $1-\theta$ denotes the "benefit "criterion. The value of the parameter θ is provided by the DM in advance. The proposed ranking index is intelligible and intrinsically superior to the original ranking index in seeking compromised solutions.

Step 6. Ranking of all alternatives.

The values of RCC_i are arranged in a descending order to obtain the best alternative, and we can rank all the alternatives with RCC_i.

6. Illustrative example

In this section, we employ a HFLTS evaluation model of movies by applying the proposed MADM approach, and give an example to demonstrate its validity and effectiveness.

6.1 Background

Over the past several decades, along with the high speed of economic and technology development, online shopping has become an important part of our daily life, primarily those of us who are busies, and it has also been recognized as an important approach to improve quality of life. More and more people shop online, however, the abundance of goods recommender systems information published online every day through different channels make it challenging for consumers to locate the content they are interested in. Recommender systems have shown to be a valuable tool to help users in such situations of information overload. The main tasks of such systems are typically to filter incoming streams of information according to the users' preferences or to point them to additional items of interest in the context of a given object. Recommenders have been successfully applied in a variety of domains, and the recommendable objects include movies, books, travel and tourism services, research articles, search queries, and so on.

Next, we would like to employ an illustrative example to provide certain reference value for linguistic evaluation model of movies by applying the proposed MADM approach.

6.2 Case study

Flimmit, is a video on demand platform which enables film lovers to obtain movies legally on the internet to enrich user experience with personalized recommendations. When viewing a movie details page, alternative movies based on other user's viewing habits are suggested. Suppose that there is a panel with five hot movies is playing to rave reviews, they are x_1: "Jurassic World: Fallen Kingdom", x_2: "A strong insect crossing the river", x_3: "The Incredibles 2", x_4: "Lobster Cop" and x_5: "Hello My Dog". The movie recommender system gives their rating according to the four attributes: Music (c_1), Frames (c_2), Director (c_3) and Story(c_4). Due to the highly-unstructured characteristics of this evaluation activity, assessment values are hardly to be assigned for the drastic fluctuations of box office returns and the moviegoers are inclined to be hesitant or irresolute in assigning their assessments. Therefore, the film reviews are provided in terms of HFLTSs on the alternatives $x_i (i = 1, 2, 3, 4, 5)$ under the four attributes $c_j (j = 1, 2, 3, 4)$. Assume that the five alternative are to be evaluated using the following LTS $H = \{s_0 : Very\ bad, s_1 : Bad, s_2 : Somewhat\ bad, s_3 : Fair, s_4 : Somewhat\ good, s_5 : Good, s_6 : Very\ good\}$. To get more objective and reasonable evaluation results, we investigate effective MADM approach for the evaluation of movies, where weights of attributes are unknown due to problem complexity. Then, the performance linguistic evaluations for each alternative $x_i (i = 1, 2, 3, 4, 5)$ are listed in Table II.

Table 2: Hesitant fuzzy linguistic decision matrix A

	c_1	c_2	c_3	c_4
x_1	$\{s_1, s_2, s_3\}$	$\{s_3, s_4\}$	$\{s_3, s_4, s_5\}$	$\{s_4, s_5\}$
x_2	$\{s_3, s_4, s_5\}$	$\{s_4, s_5\}$	$\{s_3, s_4\}$	$\{s_3, s_4, s_5\}$
x_3	$\{s_5, s_6\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$	$\{s_5\}$
x_4	$\{s_3, s_4, s_5\}$	$\{s_2, s_3, s_4\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$
x_5	$\{s_2, s_3\}$	$\{s_3, s_4, s_5\}$	$\{s_3, s_4, s_5\}$	$\{s_3, s_4\}$

6.2.1 Procedure of movie recommendation problem based on the proposed MADM method

Firstly, the normalization of the decision matrix is needed. This step involves the following two points: (1) Normalization of attributes, for all the measured attributes are of the same type, they do not need normalization; (2) Supplementation of HFLEs in the decision matrix, with the fact that the HFLEs are of different lengths, thus the improved supplementary regulation is applied to this process. The above two points lead naturally to the normalized decision matrix $\tilde{A}$. Alternative x_1 is provided as a presentative example, where,

$$\tilde{a}_{11} = \{s_1, s_1, s_2, s_2, s_3, s_3\}; \tilde{a}_{12} = \{s_3, s_3, s_3, s_4, s_4, s_4\};$$
$$\tilde{a}_{13} = \{s_3, s_3, s_4, s_4, s_4, s_4\}; \tilde{a}_{14} = \{s_4, s_4, s_4, s_5, s_5, s_5\}.$$

The next step is the generation of attributes weights by maximizing deviation method. GDDs among all attributes can be determined by different distance measures. With different distance measures, the optimal weights for each attribute c_j are subsequently different, which are shown in Tables 3 and 4.

Table 3: The weights of attributes obtained by the generalized weighted distance measure

	$\lambda=1$	$\lambda=2$	$\lambda=4$	$\lambda=8$	$\lambda=12$	$\lambda=15$
ω_1	0.4162	0.2776	0.2154	0.1903	0.1835	0.1811
ω_2	0.2289	0.1640	0.1336	0.1213	0.1180	0.1168
ω_3	0.2085	0.3280	0.3824	0.4043	0.4103	0.4124
ω_4	0.1465	0.2304	0.2686	0.2841	0.2882	0.2897

Table 4: The weights of attributes obtained by the generalized hybrid distance measure

	$\lambda=1$	$\lambda=2$	$\lambda=4$	$\lambda=8$	$\lambda=12$	$\lambda=15$
ω_1	0.4256	0.3450	0.2950	0.2705	0.2630	0.2603
ω_2	0.2653	0.2230	0.1952	0.1812	0.1769	0.1752
ω_3	0.1762	0.2358	0.2735	0.2920	0.2976	0.2997
ω_4	0.1329	0.1962	0.2364	0.2563	0.2625	0.2648

Next, we establish the HFLPIS and HFLNIS, which are conducted via (34)-(35) and shown as

$$V^+ = (\tilde{v}_1^+, \tilde{v}_2^+, ..., \tilde{v}_n^+) = \{\{s_6\}, \{s_6\}, \{s_6\}, \{s_5\}\},$$
$$V^- = (\tilde{v}_1^-, \tilde{v}_2^-, ..., \tilde{v}_n^-) = \{\{s_1\}, \{s_2\}, \{s_3\}, \{s_3\}\}.$$

Calculate the distance between each alternative x_i and the HFLPIS V^+, the distance between each alternative x_i and the HFLPIS V^-, respectively. The RCC for each alternative x_i can be calculated using Eq. (38), here we choose $\theta = 0.5$.

Table 5: The RCCs obtained by the generalized weighted distance measure

	x_1	x_2	x_3	x_4	x_5	Rankings
$\lambda=1$	0.4140	0.5088	0.6148	0.5331	0.4293	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$\lambda=5$	0.4369	0.5074	0.5823	0.5162	0.4573	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$\lambda=10$	0.4336	0.5142	0.5839	0.5038	0.4645	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=20$	0.4308	0.5190	0.5848	0.4975	0.4680	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=40$	0.4297	0.5216	0.5851	0.4944	0.4692	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$

Table 6: The RCCs obtained by the generalized weighted hybrid distance measure

	x_1	x_2	x_3	x_4	x_5	Rankings
$\lambda=1$	0.4220	0.5167	0.6146	0.5080	0.4388	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=5$	0.4339	0.5160	0.5887	0.5026	0.4587	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=10$	0.4316	0.5186	0.5870	0.4979	0.4649	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=20$	0.4300	0.5211	0.5861	0.4948	0.4680	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=40$	0.4293	0.5227	0.5858	0.4930	0.4693	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$

Table 7: The RCCs obtained by the generalized weighted Hausdorff distance measure

	x_1	x_2	x_3	x_4	x_5	Rankings
$\lambda=1$	0.4418	0.5118	0.5887	0.5040	0.4536	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=5$	0.4308	0.5206	0.5882	0.4978	0.4626	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=10$	0.4279	0.5223	0.5869	0.4960	0.4669	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=20$	0.4276	0.5233	0.5860	0.4941	0.4690	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
$\lambda=40$	0.4280	0.5238	0.5857	0.4927	0.4698	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$

Once the distance measures are determined, then their corresponding RCCs for each alternative x_i are subsequently obtained. With different distance measures, the RCCs are different, they are shown as Tables 5-7 respectively. Specifically, Fig 7 shows the effect on the ranking results by different settings of λ in the generalized distance measure. Observe from Fig 7, the RCCs are increasing or decreasing as the parameter λ increases. For example, the RCCs of x_1 and x_5 are monotonically increasing as the parameter increases on the whole, while the RCCs of x_3 and x_4 are monotonically decreasing on the whole. Similar results of different distance measures can be derived as well, thus the choice of the generalized parameter λ has some effect on the ranking results.

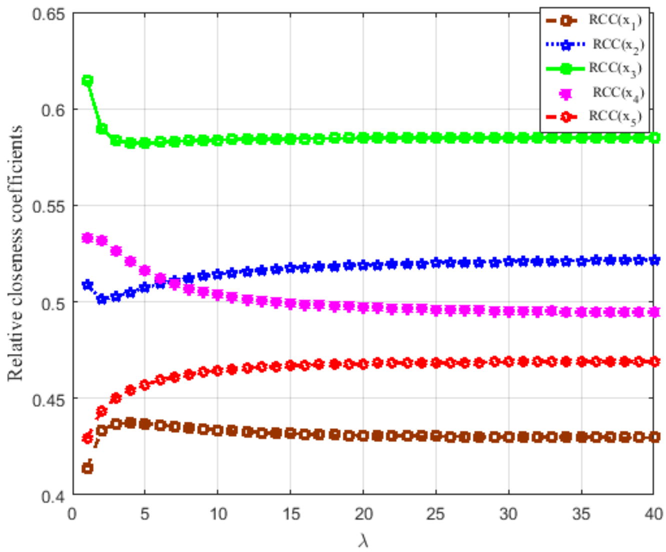

Figure 7: The RCCs obtained by the generalized weighted distance measure with different λ

From Tables 5-7, as we can see, the most desired movie is x_3: "The Incredibles 2".

Table 8: Rankings of alternatives with different assessing attitudes

Related parameters	Orness measures	Rankings
$w^1 = (0.0320, 0.0227, 0.2236, 0.3282, 0.3935)^T$	$\alpha_1 = 0.2429$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^2 = (0.0856, 0.3749, 0.3094, 0.0078, 0.2222)^T$	$\alpha_2 = 0.5235$	
$w^1 = (0.7800, 0.0248, 0.0240, 0.0317, 0.1395)^T$	$\alpha_1 = 0.8185$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^2 = (0.3676, 0.5686, 0.0524, 0.0076, 0.0037)^T$	$\alpha_2 = 0.8222$	
$w^1 = (0.1204, 0.0251, 0.0252, 0.0407, 0.7886)^T$	$\alpha_1 = 0.1620$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^2 = (0.2382, 0.0066, 0.0065, 0.0098, 0.7388)^T$	$\alpha_2 = 0.3342$	
$w^1 = (0.7800, 0.0248, 0.0240, 0.0317, 0.1395)^T$	$\alpha_1 = 0.8185$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^2 = (0.0382, 0.0066, 0.0065, 0.1098, 0.8388)^T$	$\alpha_2 = 0.0739$	
$w^1 = (0.0767, 0.6327, 0.2782, 0.0084, 0.0040)^T$	$\alpha_1 = 0.6924$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^2 = (0.3627, 0.2066, 0.1877, 0.1355, 0.1075)^T$	$\alpha_2 = 0.6454$	
$w^1 = (0.0767, 0.6327, 0.2782, 0.0084, 0.0040)^T$	$\alpha_1 = 0.6924$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^2 = (0.0382, 0.0066, 0.0065, 0.0098, 0.9388)^T$	$\alpha_2 = 0.0489$	
$w^1 = (0, 0, 0, 0, 1)^T$,$w^2 = (0, 0, 0, 0, 1)^T$	$\alpha_1 = \alpha_2 = 0$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^1 = (0, 0, 0, 0, 1)^T$,$w^2 = (1, 0, 0, 0, 0)^T$	$\alpha_1 = 0$,$\alpha_2 = 1$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^1 = (0, 0, 1, 0, 0)^T$,$w^2 = (0, 0, 1, 0, 0)^T$	$\alpha_1 = \alpha_2 = 0.5$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
$w^1 = (1, 0, 0, 0, 0)^T$,$w^2 = (1, 0, 0, 0, 0)^T$	$\alpha_1 = \alpha_2 = 1$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$

6.2.2 Sensitive analysis of the proposed NS model

Next, we would like to illustrate the assessing attitudes of DMs on the ranking results. Note that the introduced NS influences not only on the DM's NVs about the input argument information, but also on the attributes weights. With different NSs reflect different assessing attitudes towards linguistic preference, different RCCs can be obtained, ranking results with different assessing attitudes are shown in Table 8.

Thus, the most desired movie is still is x_3: "The Incredibles 2". Table 8 shows no matter what the changing of parameter is, the best alternative is always x_3. Note that it is only suitable for this example because the calculation of the most desired movie involves other factors such as the values of attributes, weights of attributes, the relatively important degree of total deviation, and the selection process, etc. Therefore, the DM's assessing attitudes have a significant effect on the calculation, and the results are consistent with practical problem.

6.2.3 Sensitive analysis of the TOPSIS index

In this subsection, we would like to discuss how the parameter θ in the proposed TOPSIS index affects the calculation results and final orderings. We have indicated the value of θ in Eq. (38) represents a decision mechanism. In practice, the parameter θ is based on the DMs' preference. Here, θ is chosen 0 to 1 increasing by 0.1 to analyse the sensitivity. Three typical distance measures are presented in Table 9.

Table 9: Sensitive analysis of parameter θ for the TOPSIS index

Distance measures	v	RCC_1	RCC_2	RCC_3	RCC_4	RCC_5	Final orderings
Weighted distance	0.0	0.1191	0.2083	0.3080	0.2311	0.1334	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.1	0.1781	0.2684	0.3694	0.2915	0.1926	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.2	0.2371	0.3285	0.4307	0.3519	0.2518	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.3	0.2960	0.3886	0.4921	0.4123	0.3109	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.4	0.3550	0.4487	0.5535	0.4727	0.3701	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.5	0.4140	0.5088	0.6148	0.5331	0.4293	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.6	0.4730	0.5689	0.6762	0.5935	0.4884	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.7	0.5320	0.6291	0.7375	0.6538	0.5476	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.8	0.5910	0.6892	0.7989	0.7142	0.6068	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.9	0.6500	0.7493	0.8602	0.7746	0.6659	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	1.0	0.7089	0.8094	0.9216	0.8350	0.7251	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
Weighted Hausdorff distance	0.0	0.1461	0.2098	0.2709	0.2179	0.1554	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.1	0.2052	0.2702	0.3344	0.2752	0.2150	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.2	0.2644	0.3306	0.3980	0.3324	0.2747	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
	0.3	0.3235	0.3910	0.4616	0.3896	0.3343	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.4	0.3827	0.4515	0.5251	0.4468	0.3940	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.5	0.4418	0.5118	0.5887	0.5040	0.4536	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.6	0.5010	0.5722	0.6523	0.5612	0.5133	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.7	0.5601	0.6326	0.7159	0.6185	0.5729	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.8	0.6193	0.6930	0.7794	0.6757	0.6326	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.9	0.6784	0.7535	0.8430	0.7329	0.6922	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	1.0	0.7376	0.8139	0.9066	0.7901	0.7519	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
Weighted hybrid distance	0.0	0.1255	0.2160	0.3072	0.2102	0.1412	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.1	0.1848	0.2761	0.3687	0.2697	0.2007	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.2	0.2441	0.3362	0.4302	0.3293	0.2602	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.3	0.3034	0.3964	0.4916	0.3888	0.3197	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.4	0.3627	0.4565	0.5531	0.4484	0.3792	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.5	0.4220	0.5167	0.6146	0.5080	0.4388	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.6	0.4813	0.5768	0.6761	0.5675	0.4983	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.7	0.5406	0.6369	0.7376	0.6271	0.5578	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.8	0.5999	0.6971	0.7991	0.6866	0.6173	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	0.9	0.6592	0.7572	0.8606	0.7462	0.6768	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$
	1.0	0.7185	0.8173	0.9221	0.8057	0.7363	$x_3 \succ x_2 \succ x_4 \succ x_5 \succ x_1$

From Table 9, by different settings of distance measures and θ, it seems that the most desired movie is still x_3. The result indicates that our proposed method is robust and reliable.

6.3 Comparative analysis

6.3.1 Comparative analysis with existing distance measures for HFLTSs

To demonstrate the feasibility and applicability of our proposed distance measures for HFLTSs, we use some existing distance measures [41, 44] for comparison analysis. The distance measures in [44] have the similar form as Eqs. (23)-(25), the weighted cosine distance measure in [41] is given as

$$d_{WCD}(H_S^1, H_S^2) = 1 - \frac{\sum_{j=1}^{n} \frac{\omega_j}{L_j} \sum_{l=1}^{L_j} \left(\frac{|\theta_l^1(x_j)|}{2\tau+1} \cdot \frac{|\theta_l^2(x_j)|}{2\tau+1} \right)}{\left(\sum_{j=1}^{n} \left(\frac{\omega_j}{L_j} \sum_{l=1}^{L_j} \left(\frac{\theta_l^1(x_j)}{2\tau+1} \right)^2 \right) \cdot \sum_{j=1}^{n} \left(\frac{\omega_j}{L_j} \sum_{l=1}^{L_j} \left(\frac{\theta_l^1(x_j)}{2\tau+1} \right)^2 \right) \right)^{1/2}}. \quad (39)$$

An essential difference between these distance measures and new distance measures in this paper is the way on dealing with HFLEs that are of different lengths. In existing methods, the short HFLEs are extended by adding the LT $s_\theta = (s_\theta^+ \oplus s_\theta^-)/2$. We now use the existing generalized weighted distance and weighted cosine distance measures of HFLTSs for comparison analysis, the rest can be analyzed similarly.

Firstly, the generalized weighted distance measure in [44] is taken into consideration. The results are shown in Fig. 8. According to Fig. 8, we observe the ranking results are different, thus leading to different decisions. However, it seems alternative x_3 is always the best one(s). Meanwhile, the ranking results obtained by Liao et al. [44] are not exactly the same as the proposed method, the reason is that the shorter HFLEs are extended by some artificial values, this may lead to the initial information distortion and losing, which has significant influence on the calculation.

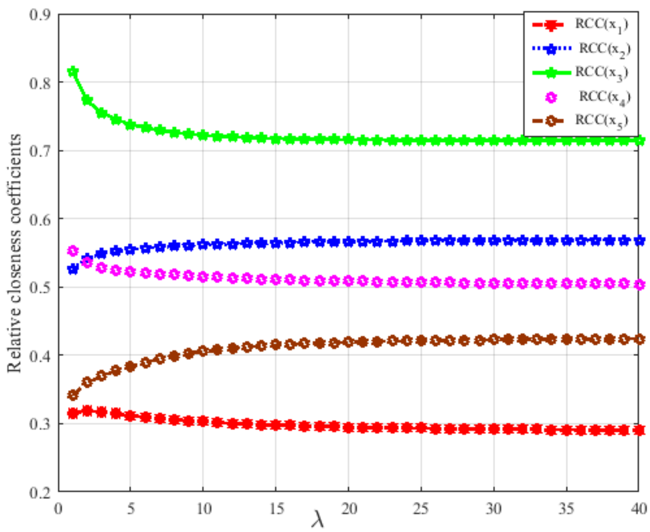

Figure 8: The RCCs obtained by the generalized weighted distance measure with different λ

In the following, we use the cosine-distance-based HFL-TOPSIS method to solve the same illustrative example, the decision steps are as follows:

Step 1. According to the score function and dispersion degree for each HFLE, we have $\max(c_1) = h_S^{31} = \{s_5, s_6\}, \min(c_1) = h_S^{11} = \{s_1, s_2, s_3\}, \max(c_2) = h_S^{33} = \{s_4, s_5, s_6\}, \min(c_2) = h_S^{42} = \{s_2, s_3, s_4\}, \max(c_3) = h_S^{43} = \{s_4, s_5, s_6\}, \min(c_3) = h_S^{32} = \{s_3, s_4\}, \max(c_4) = h_S^{43} = \{s_5\}, \min(c_4) = h_S^{54} = \{s_3, s_4\}$. Thus, the HFL-PIS and the HFL-NIS for this problem are obtained

$$V^{+} = \{\{s_5, s_6\}, \{s_4, s_5, s_6\}, \{s_4, s_5, s_6\}, \{s_5\}\},$$
$$V^{-} = \{\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \{s_3, s_4\}, \{s_3, s_4\}\}.$$

Step 2. The relative closeness (RC) of each alternative can be obtained as $RC(x_1) = 0.4251, RC(x_2) = 0.7969, RC(x_3) = 0.9683, RC(x_4) = 0.7834$ and $RC(x_5) = 0.5192$.
Step 3. According to the values of RC of each alternative, alternative x_3: "The Incredibles 2" is the best choice.

Distance measures proposed in [41] are based on the geometric point of view. Hence, the distance measures proposed in this paper are quite different from them. The cosine distance measures are much easier to be understood because their geometric meanings are quite intuitive. If we select two HFLTSs $\mathrm{H}_S^1 = \{s_{-3}\}$ and $\mathrm{H}_S^2 = \{s_3\}$, according to Eq. (39), the cosine-distance-based measure between H_S^1 and H_S^2 is 0, it is apparently contrast with our thought analysis. Therefore, this distance measure still has its own weaknesses.

According to the comparison analyses above, the distance measures in this paper offer many advantages, which can be concluded as follows:

- The proposed NS model transforms LTs into NVs is more flexible to reflect the assessing attitudes of DMs towards linguistic preferences. The decision making method is shown to be more suitable for application to the MADM process involving PIS than the normal linguistic MADM methods, which demonstrates the applicability of the new method in real decision making problems.

- The primary advantage of the improved supplementary regulation for HFLEs is that it is not affected by subjective factors of DMs, which can satisfactorily avoid the initial information distortion and losing in the linguistic information processing. On the basis of the improved supplementary regulation, some important measures such as the distance measures, entropy measures and so on can be redefined and applied to practical problems.

- The prominent characteristics of the distance measures for HFLTSs are not only for its effectiveness in dealing with the preference information expressed by LTs, but also for that they provide a very general formula including a wide range of distance measures, which make the final results in line with the real decision making problems.

6.3.2 Comparative analysis with existing NS models

To further illustrate the advantages of the proposed NS model, some existing NS models [8,31,39-41] are proposed for comparison analysis. The ranking results are shown in Table 9.

Table 10: Rankings obtained by some existing NS models.

NS models	Basic NS functions	Rankings
Xu [31]	$f(x) = kx$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
Wang et al. [29]	$f(x) = x^{\alpha}$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
Bao et al. [37]	$f(x) = a^{kx+b}$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
Zhou and Xu [38]	$f(x) = \frac{1}{1+e^{-kx}}$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$
Liu and Liu [40]	$f(x) = x^{\alpha}$	$x_3 \succ x_4 \succ x_2 \succ x_5 \succ x_1$

Obviously, our method can obtain different ranking results for PIS situations, while the NSs above may only represent a specific semantic. Compared with [8,31,39-41], some advantages of the proposed NS model are shown as

- The proposed NS model generalizes several existing NS approaches, and it can be used to express LTSs that are not uniformly and symmetrically distributed naturally.
- Our proposed NS model is based on the LTs with their corresponding NVs provided by each individual, it is a natural premise which can be achieved easily. It can satisfactorily denote the concavity and convexity of the subjective NSs for inferior LTs and superior LTs respectively.
- Our linguistic model utilizes the fractional polynomial function to approximate the NS model, when there is not too many data for fitting, polynomial curve function can acquire good fitting effect.
- The parameters in our NS model can be solved by a linear programming model. It is well known that linear programming models can be effectively solved in very little computational time.
- Unlike the other NS models which firstly determine the functional forms in their NSs models, our model firstly obtains the optimal parameters by the ODF technology, and then utilizes the 'orness' measure to quantitatively describe the assessing attitudes.

7. Conclusion

In this paper, we provide a specific numerical scale model with the purpose of making transformations between linguistic terms and numerical values. The proposed linguistic model can represent a wide range of linguistic behaviours and increase the flexibility of the linguistic information. We establish an improved supplementary regulation for HFLTS to achieve the highest level of retention of information. Based on some traditional distance measures, some new distance measures for HFLEs and HFLTSs are provided. Furthermore, an extended TOPSIS method based on proposed distance measures for hesitant fuzzy linguistic MADM is developed.

In the future research, we would like to mainly research the linguistic representation model on the following points: (1) In practical life, if DMs express their opinions via other qualitative natural linguistic preference structures, how to handle such information with numerical scale model is wondered; (2) The individual linguistic behavior may be approximated satisfactorily by NS models, while how to approximate and simulate the group linguistic behaviors is still questionable; (3) The existing linguistic models are quite fruitful, it is necessary to propose some criteria to compare them.

Besides, we can extend the application scopes of the proposed methods to different decision making fields such as option of supply chain [49], industrial 5.0 [50], digital twin [51], and so on [52–54].

Acknowledgement

The work was supported by National Natural Science Foundation of China (Nos. 72301003, 72171002, U22A20366, 72201004), Research Funding Project of Academic and technical leaders and reserve candidates in Anhui Province (No.2018H179), Top Talent Academic Foundation for University Discipline of Anhui Province (No.gxbjZD2020056), Excellent University Research and Innovation Teams in Anhui Province (No. 2024AH010002), College Student Innovation and Entrepreneurship Training Program of Anhui Province (No. S202410357009).

Conflicts of Interest

The authors declare no conflicts of interest.

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